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On the functional equivalence of two perfectly competitive economies with negative exponential utility and linear utility with a quadratic holding cost

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Abstract
In this paper, we analyze the functional equivalence of two perfectly competitive economies with negative exponential and linear utility with a quadratic holding cost. The two economies are said to be functionally equivalent if there exists a one-to-one correspondence between the vector of holding costs and the vector of risk aversion coefficients such that the resulting two economies have the same market equilibrium. If the information was symmetric, the equilibrium price reveals no new information and the functional equivalence between the two economies is straightforward. However, in the case of asymmetric information, the equilibrium price reveals some new information and an endogeneity issue arises. We establish the functional equivalence between the two economies with asymmetric information by resolving this endogeneity problem through a fixed-point argument.

Keywords Functional equivalency · Negative exponential utility · Linear utility with a quadratic holding cost

Mathematics Subject Classification D82 · G14

1 Introduction

The literature on rational expectations equilibrium (REE) and Bayesian market games typically has two classes of utility functions: negative exponential utility, for example, see [1–6], and linear utility with a quadratic holding cost, for example, see [7–11]. In this paper, we

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analyze the functional equivalence between the two classes of utility functions in perfectly competitive markets. In the model, there is one risky asset and one risk-less asset traded by a continuum of agents. Each trader can also observe a private signal that is correlated with the fundamental of the risky asset, besides the public information of the price. To prevent prices from being fully revealing, the model includes noise trading. The equilibrium price is endogenously determined by market-clearing conditions. We analyze the equivalence between the two economies that are identical except that one is equipped with negative exponential utility and the other one is equipped with linear utility with a quadratic holding cost, which are referred to as CARA economy and quadratic economy, respectively.

Under the common assumption of Gaussian random variables, we show that the CARA economy and the quadratic economy are *functionally* equivalent; that is, there exists a one-to-one correspondence between the vector of holding costs and the vector of agents' risk aversion coefficients such that the resulting two perfectly competitive economies yield the same market equilibrium. Our equivalence result reveals that the choice of negative exponential and linear utility with a quadratic holding cost is not crucial in classical REE models because these classes of utility functions yield the same set of market equilibrium outcomes, and the behavior of traders in one economy could be exactly replicated by properly adjusting the risk aversion/holding cost parameters in the other economy. This research also deepens our understanding of the historical disagreement over the use of negative exponential and quadratic utility to model and predict agents' behavior [12].

If the information was symmetric, the equilibrium price does not reveal any new information and then conditioning on equilibrium prices becomes superfluous. In this case, the functional equivalence between the CARA economy and quadratic economy is trivial and boils down to the well-known equivalence between CARA-normal distribution preferences and mean-variance preferences. The situation is different for the case of asymmetric information considered here where equilibrium prices reveal in a noisy REE some new information. Because equilibrium prices in CARA economies are now endogenously determined (as the expectation and variance are now conditioned on prices), an endogeneity issue arises. As a consequence of this endogeneity issue, it is no longer obvious that each noisy REE in the quadratic economy can be replicated by some (unique) noisy REE in the CARA economy. This paper's main contribution is to address and resolve this endogeneity problem through a fixed-point argument (cf. the proof of Part (ii) of Proposition 1).

2 Problem formulation

This is a single-period model. All random variables in the model are normally distributed and the means are normalized to zero without loss of generality. The economy is populated by a continuum of agents with unit mass. The economy has one riskless asset that pays zero interest with perfectly elastic supply, and one risky asset that has a payoff of $\theta \sim N(0, 1/\tau_\theta)$, $\tau_\theta > 0$. Agents cannot directly observe the fundamental θ at the beginning of the period; instead, each agent $i \in [0, 1]$ can observe a private signal $y_i = \theta + \varepsilon_i$, where $\varepsilon_i \sim N(0, 1/\tau_i)$, $\tau_i > 0$. To prevent prices from being fully revealing, the economy includes noise demand $u \sim N(0, 1/\tau_u)$, $\tau_u > 0$. Let x_i denote the holdings of the risky asset by agent i. The wealth of agent i at the end of the investment period is then given by $W_{i,1} = x_i(\theta - p)$, where p is the price of the risky asset, and here we assume without loss of generality that all agents' initial wealth is zero.

We next introduce two identical economies, which have the same fundamental θ of the risky asset, agents' private signals $\{y_i\}_i$, and noise demand u, except that agents have different utility functions. In this paper, we use the superscripts C and Q to denote the CARA economy with negative exponential utility and the quadratic economy in which traders have a linear utility with a quadratic holding cost, respectively.

The CARA Economy E^C. The terminal utility agent i obtains from taking demand x_i, $U_i^C(x_i)$, is

$$U_i^C(x_i) = -\exp\{-\rho_i x_i(\theta - p)\},$$

where $\rho_i > 0$ is agent i's constant absolute risk aversion (CARA) coefficient. In economy E^C, agents are risk averse with a CARA coefficient. Given its mathematical tractability, prior studies on REE commonly adopt negative exponential utility, for example, [1–6].

The Quadratic Economy E^Q **with Linear Utility and a Quadratic Holding Cost**. The terminal utility agent i obtains from taking demand x_i, $U_i^Q(x_i)$, is

$$U_i^Q(x_i) = (\theta - p)x_i - \frac{\phi_i}{2}x_i^2,$$

where $\phi_i > 0$ represents agent i's holding cost. In economy E^Q, agents are risk neutral, but with a quadratic holding cost. For its simple form, prior works also typically adopt linear utility with a quadratic holding cost, for example, [7–11].

A *noisy REE* is a tuple $(x_i, i \in [0, 1], p)$ of agents' optimal demand decisions and price such that (i) the conditional expectation of agent i's utility is maximized at x_i; that is, $x_i \in \arg\max_x \mathbb{E}[U_i^{\cdot}(x)|y_i, p]$; and (ii) the market clears; that is, $\int_0^1 x_i di + u = 0$ almost surely. In a noisy REE, traders make their demand decisions optimally based on their own private information and the information contained in the price of the risky asset, where the price is the one at which the market clears. The price in a noisy REE is called the equilibrium price. The main aim of this paper is to show that the two economies E^C and E^Q are *functionally equivalent*, that is, there exists a one-to-one correspondence between $\{\phi_i\}_i$ of the holding costs and $\{\rho_i\}_i$ of the risk aversion coefficients such that the resulting two economies E^C and E^Q have the same market equilibrium.

3 Main results

Before presenting the main result, we make a technical assumption on the value of signal precisions, risk aversion, and holding costs to facilitate the analysis.

Assumption 1 There are a finite number of types of combinations of signal precisions and risk aversion coefficients $\{(\rho_1^\diamond, \tau_1^\diamond), \ldots, (\rho_m^\diamond, \tau_m^\diamond)\}$, a finite number of types of combinations of signal precisions and holding costs $\{(\phi_1^\diamond, \tau_1^\diamond), \ldots, (\phi_m^\diamond, \tau_m^\diamond)\}$, and a fraction $0 < \lambda_k < 1$ of type-k traders with $\sum_{k=1}^m \lambda_k = 1$ such that

- In economy E^C, $(\rho_i, \tau_i) \in \{(\rho_1^\diamond, \tau_1^\diamond), \ldots, (\rho_m^\diamond, \tau_m^\diamond)\}$ for any $i \in [0, 1]$;
- In economy E^Q, $(\phi_i, \tau_i) \in \{(\phi_1^\diamond, \tau_1^\diamond), \ldots, (\phi_m^\diamond, \tau_m^\diamond)\}$ for any $i \in [0, 1]$.

Following the standard procedures for solving linear equilibrium in the literature [2, 13], we can show the existence and uniqueness of a linear noisy REE in economy E^C, and the equilibrium price is

$$p = \frac{1}{\Delta + \frac{\tau_\theta}{\Delta\tau_u + \rho}}(\Delta\theta + u), \tag{1}$$

and the optimal demand decision of trader i equals

$$x_i(y_i, p) = \frac{\mathbb{E}[\theta|y_i, p] - p}{\rho_i \operatorname{Var}[\theta|y_i, p]} = \rho_i^{-1}\tau_i y_i - \left(\rho_i^{-1}\tau_i + \frac{\tau_\theta}{\Delta\tau_u + \rho}\right)p, \tag{2}$$

where

$$\rho = \left(\sum_{k=1}^{m} \frac{\lambda_k}{\rho_k^\diamond}\right)^{-1},$$

and

$$\Delta = \int_0^1 \frac{\tau_i}{\rho_i} di = \sum_{k=1}^{m} \lambda_k \frac{\tau_k^\diamond}{\rho_k^\diamond}. \tag{3}$$

Clearly, in the linear noisy REE in economy E^Q, trader i's optimal demand decision equals

$$x_i(y_i, p) = \frac{\mathbb{E}[\theta|y_i, p] - p}{\phi_i}, i \in [0, 1]. \tag{4}$$

Here, we temporarily do not attempt to solve the market equilibrium in economy E^Q, and instead first assume that a linear noisy REE exists. In fact, as a byproduct of the following proposition, the linear noisy REE indeed exists.

Let $\boldsymbol{\phi} = (\phi_1^\diamond, \ldots, \phi_m^\diamond)$, $\boldsymbol{\rho} = (\rho_1^\diamond, \ldots, \rho_m^\diamond)$. From (2) and (4), we see that to establish the functional equivalence between economy E^C and economy E^Q, it is crucial to show that for each $\boldsymbol{\phi}$ of holding costs, there exists a unique $\boldsymbol{\rho}$ of risk aversion coefficients such that $\rho_k^\diamond \operatorname{Var}[\theta|y_i, p] = \phi_k^\diamond$ for each agent i with risk aversion $\rho_k^\diamond$ and signal precision $\tau_k^\diamond$. It seems sufficient to set $\rho_k^\diamond = \phi_k^\diamond / \operatorname{Var}[\theta|y_i, p]$, but we should note that the equilibrium price p depends on the risk aversion $\boldsymbol{\rho}$, as (1) indicates. Hence, showing the functional equivalency is not straightforward. The following is the main result of this paper.

Proposition 1 *Suppose* Assumption 1 *holds. We have*

(i) *If $(x_i, i \in [0, 1], p)$ is a noisy REE for the CARA economy $E^C(\boldsymbol{\rho})$, then there exists a unique $\boldsymbol{\phi}$ such that $(x_i, i \in [0, 1], p)$ is also a noisy REE for the quadratic economy $E^Q(\boldsymbol{\phi})$ whereby*

$$\phi_i = \rho_i \operatorname{Var}[\theta|y_i, p] \;\; for \;\; all \;\; i.$$

(ii) *If $(x_i, i \in [0, 1], p)$ is a noisy REE for the quadratic economy $E^Q(\boldsymbol{\phi})$, then there exists a unique $\boldsymbol{\rho}$ such that $(x_i, i \in [0, 1], p)$ is also a noisy REE for the CARA economy $E^C(\boldsymbol{\rho})$ whereby $\boldsymbol{\rho}$ satisfies*

$$\rho_i \operatorname{Var}[\theta|y_i, p(\cdot, \boldsymbol{\rho})] = \phi_i \;\; for \;\; all \;\; i.$$

Proof Part (i) is straightforward with the parameter setting $\phi_k^\diamond = \rho_k^\diamond \operatorname{Var}[\theta|y_i, p]$ for any agent i with risk aversion $\rho_k^\diamond$ and signal precision $\tau_k^\diamond$. We next show part (ii). Recalling Assumption 1, it suffices to show that for any given $\boldsymbol{\phi} = (\phi_1^\diamond, \ldots, \phi_m^\diamond) \in \mathbb{R}_{++}^m$, there exists a unique $\boldsymbol{\rho} = (\rho_1^\diamond, \ldots, \rho_m^\diamond) \in \mathbb{R}_{++}^m$ such that the following equations hold:

$$\rho_k^\diamond \operatorname{Var}[\theta|y_i, p(\cdot, \boldsymbol{\rho})] = \frac{\rho_k^\diamond}{\tau_\theta + \tau_k^\diamond + (\Delta(\boldsymbol{\rho}))^2\tau_u} = \phi_k^\diamond, k = 1, \ldots, m, \tag{5}$$

where i is one type-k agent with risk aversion $\rho_k^\diamond$ and signal precision $\tau_k^\diamond$, the first equality follows from (1) and the projection theorem for normal random variables, and $\Delta(\boldsymbol{\rho}) = \sum_{k=1}^m \lambda_k \tau_k^\diamond / \rho_k^\diamond$ is a function of $\boldsymbol{\rho} = (\rho_1^\diamond, \ldots, \rho_m^\diamond) \in \mathbb{R}_{++}^m$; see (3). In the remaining proofs, we omit the superscript $\diamond$ for notational convenience.

We first show the existence. Define

$$q_k(\boldsymbol{\rho}) := (\tau_\theta + \tau_k + (\Delta(\boldsymbol{\rho}))^2 \tau_u)\phi_k,\, k = 1, \ldots, m.$$

We can see that $q_k(\boldsymbol{\rho}) \geq (\tau_\theta + \tau_k)\phi_k$ for any $\boldsymbol{\rho} \in \mathbb{R}_{++}^m$, and

$$q_k(\boldsymbol{\rho}) \leq \left(\tau_\theta + \tau_k + \left(\frac{\sum_{k=1}^m \lambda_k \tau_k}{\min_{1\leq k\leq m}[(\tau_\theta + \tau_k)\phi_k]}\right)^2 \tau_u\right)\phi_k$$

for any $\boldsymbol{\rho} \in \mathbb{R}_{++}^m$ with $\rho_k \geq (\tau_\theta + \tau_k)\phi_k$ for every k. Thus, $(q_1, \ldots, q_m)$ maps the convex compact set

$$\prod_{k=1}^m \left[(\tau_\theta + \tau_k)\phi_k, \left(\tau_\theta + \tau_k + \left(\frac{\sum_{k=1}^m \lambda_k \tau_k}{\min_{1\leq k\leq m}[(\tau_\theta + \tau_k)\phi_k]}\right)^2 \tau_u\right)\phi_k\right]$$

onto itself. Moreover, $(q_1, \ldots, q_m)$ is continuous. It follows from Brouwer's fixed point theorem that the mapping $(q_1, \ldots, q_m)$ has a fixed point, or equivalently, (5) has a solution $(\rho_1, \ldots, \rho_m) \in \mathbb{R}_{++}^m$.

We now show the uniqueness of the solution to (5). We first show that the averaged level of risk seeking $\sum_{k=1}^m \lambda_k/\rho_k$ is uniquely determined by the given parameters $\{\phi_k, k = 1, \ldots, m\}$. Let $(\rho_1, \ldots, \rho_m)$ be a solution to (5). From (5) we have

$$\frac{\tau_\theta + \Delta^2\tau_u}{\rho_k} + \frac{\tau_k}{\rho_k} = \frac{1}{\phi_k},\, k = 1, \ldots, m. \tag{6}$$

Multiplying both sides of (6) by λ_k, taking the sum over k, and noting the definition of Δ, we have

$$\left(\sum_{k=1}^m \frac{\lambda_k}{\rho_k}\right)(\tau_\theta + \Delta^2\tau_u) + \Delta = \sum_{k=1}^m \frac{\lambda_k}{\phi_k}. \tag{7}$$

Corresponding to (7), we introduce an auxiliary equation,

$$t(\tau_\theta + w_t^2\tau_u) + w_t = \sum_{k=1}^m \frac{\lambda_k}{\phi_k}, \tag{8}$$

where w_t is a function of variable t such that (8) holds. Taking the derivative of both sides of (8) with respect to t yields

$$\tau_\theta + w_t^2\tau_u + 2tw_t\tau_u\frac{\partial w_t}{\partial t} + \frac{\partial w_t}{\partial t} = 0,$$

from which

$$\frac{\partial w_t}{\partial t} = -\frac{\tau_\theta + w_t^2\tau_u}{2tw_t\tau_u + 1}.$$

Moreover, from (6) we also alternatively have

$$\frac{1}{\rho_k} = \frac{1}{\phi_k(\tau_\theta + \tau_k + \Delta^2\tau_u)}.$$

Multiplying both sides of the preceding equation by λ_k and taking the sum over k gives

$$\sum_{k=1}^{m} \frac{\lambda_k}{\rho_k} = \sum_{k=1}^{m} \frac{\lambda_k}{\phi_k(\tau_\theta + \tau_k + \Delta^2 \tau_u)}.$$

Corresponding to the above equation, we define an auxiliary function

$$h(t) := t - \sum_{k=1}^{m} \frac{\lambda_k}{\phi_k(\tau_\theta + \tau_k + w_t^2 \tau_u)}.$$

We next show that the solution t for the equation $h(\cdot) = 0$ is unique. In fact, we have

$$\begin{aligned} \frac{\partial h(t)}{\partial t} &= 1 + 2w_t \tau_u \sum_{k=1}^{m} \frac{\lambda_k \frac{\partial w_t}{\partial t}}{\phi_k(\tau_\theta + \tau_k + w_t^2 \tau_u)^2} \\ &= 1 - \frac{2w_t \tau_u(\tau_\theta + w_t^2 \tau_u)}{2t w_t \tau_u + 1} \sum_{k=1}^{m} \frac{\lambda_k}{\phi_k(\tau_\theta + \tau_k + w_t^2 \tau_u)^2} \\ &> 1 - \frac{2w_t \tau_u}{2t w_t \tau_u + 1} \sum_{k=1}^{m} \frac{\lambda_k}{\phi_k(\tau_\theta + \tau_k + w_t^2 \tau_u)} \\ &= 1 - \frac{2t w_t \tau_u}{2t w_t \tau_u + 1} > 0. \end{aligned}$$

We also have $h(0) < 0$ and $\lim_{t\to\infty} h(t) = \infty$ because $\lim_{t\to\infty} w_t = 0$. Hence, the solution t to the equation $h(\cdot) = 0$ is unique. We then can see that, for any solution $(\rho_1, \ldots, \rho_m)$ (if many), the variable $\sum_{k=1}^{m} \lambda_k/\rho_k$ is a constant. Moreover, the uniqueness of the solution w_t to (8) for any $t > 0$ implies that $\Delta = \sum_{k=1}^{m} \lambda_k \tau_k/\rho_k$ is also a constant, denoted as $\hat{\Delta}$, for any solution to (5). Therefore, $(\rho_1, \ldots, \rho_m)$ is uniquely determined by $\rho_k = \phi_k(\tau_\theta + \tau_k + \hat{\Delta}^2 \tau_u)$, $k = 1, \ldots, m$. This completes the proof. □

Declarations

Conflict of interest The author declares that they have no conflict of interest.

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