# Online Supplemental Material for "Information Sharing in a Perfectly Competitive Economy"

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In this Supplemental Material we provide the proofs of the welfare effects of network connectedness in Subsection 4.2 and Subsection 4.3. The numbers for equations correspond to those in the main paper, unless they are specific to the Supplemental Material, in which case the numbers are prefixed by an "S".

## S1 The Welfare Effects in Subsection 4.2

Here we provide the detailed proofs of the conclusions on the welfare effects in Subsection 4.2. We still analyze the certainty equivalent instead of the welfare. From (A.1), we have

$$\begin{aligned}\mathbb{E}\left[U(\tilde{d}^*_{i,g}(v-\tilde{p})-c(\tilde{\rho}_{\epsilon,k}))\right] &= -e^{\gamma c(\tilde{\rho}_{\epsilon,k})}\sqrt{\frac{1}{\rho_v+\tilde{\rho}_\theta+\rho_\zeta+k\tilde{\rho}_{\epsilon,k}}\frac{1}{(1-\tilde{\alpha}_v)^2\rho_v^{-1}+\tilde{\alpha}_x^2\rho_x^{-1}}} \\ &= -e^{\gamma c(\tilde{\rho}_{\epsilon,k})}\sqrt{1-\frac{\gamma^4\rho_x^{-1}+\gamma^2\tilde{z}_k}{\rho_x\tilde{z}_k^2+2\gamma^2\tilde{z}_k+\gamma^2\rho_v+\gamma^4\rho_x^{-1}}}, \end{aligned} \tag{S1}$$

where $\tilde{d}^*_{i,g}$ refers to the optimal demand of trader $i$ in group $g$. The first equality follows from (4) and the projection theorem for normal random variables, and (S1) follows from the expressions of $\tilde{\alpha}_v$, $\tilde{\alpha}_x$ and $\tilde{\rho}_\theta$ defined by (5), (6) and (7). Then, from (S1), the certainty equivalent in the

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presence of public information equals

$$\widetilde{CE}(k) := -\frac{1}{\gamma}\log\left[\mathbb{E}(\exp\{-\gamma\tilde{d}_{i,g}^{*}(v-p)\})\right] - c(\tilde{\rho}_{\epsilon,k})$$
$$= \frac{1}{2\gamma}\log\left[\frac{\rho_x\tilde{z}_k^2 + 2\gamma^2\tilde{z}_k + \gamma^2\rho_v + \gamma^4\rho_x^{-1}}{\rho_x\tilde{z}_k^2 + \gamma^2\tilde{z}_k + \gamma^2\rho_v}\right] - c(\tilde{\rho}_{\epsilon,k}). \tag{S2}$$

We next show that $\partial\widetilde{CE}(k)/\partial k \propto \tilde{A}(k)$ and $\tilde{A}(k)$ is strictly decreasing in $k$. From (8), we can rewrite (S2) as

$$\widetilde{CE}(k) = \left[\frac{1}{2\gamma}\log\left(\rho_x\tilde{z}_k^2 + 2\gamma^2\tilde{z}_k + \gamma^2\rho_v + \gamma^4\rho_x^{-1}\right)\right] - \left[\frac{1}{2\gamma}\log\left(\frac{\gamma}{2c'(\tilde{\rho}_{\epsilon,k})}\right) + c(\tilde{\rho}_{\epsilon,k})\right]$$
$$=: \tilde{F}_1(k) - \tilde{F}_2(k).$$

Then,

$$\frac{\partial\widetilde{CE}(k)}{\partial k}$$
$$= \frac{\partial\tilde{F}_1(k)}{\partial\tilde{z}_k}\frac{\partial\tilde{z}_k}{\partial k} - \frac{\partial\tilde{F}_2(k)}{\partial\tilde{\rho}_{\epsilon,k}}\frac{\partial\tilde{\rho}_{\epsilon,k}}{\partial k}$$
$$= \frac{1}{2\gamma}\frac{2\rho_x\tilde{z}_k + 2\gamma^2}{\rho_x\tilde{z}_k^2 + 2\gamma^2\tilde{z}_k + \gamma^2\rho_v + \gamma^4\rho_x^{-1}}\frac{\partial\tilde{z}_k}{\partial k} - \left[-\frac{c''(\tilde{\rho}_{\epsilon,k})}{2\gamma c'(\tilde{\rho}_{\epsilon,k})} + c'(\tilde{\rho}_{\epsilon,k})\right]\frac{\partial\tilde{\rho}_{\epsilon,k}}{\partial k}$$
$$\stackrel{(9)}{=} \frac{1}{2\gamma}\frac{2\rho_x\tilde{z}_k + 2\gamma^2}{\rho_x\tilde{z}_k^2 + 2\gamma^2\tilde{z}_k + \gamma^2\rho_v + \gamma^4\rho_x^{-1}}\left(\tilde{\rho}_{\epsilon,k} + k\frac{\partial\tilde{\rho}_{\epsilon,k}}{\partial k}\right) - \left[-\frac{c''(\tilde{\rho}_{\epsilon,k})}{2\gamma c'(\tilde{\rho}_{\epsilon,k})} + c'(\tilde{\rho}_{\epsilon,k})\right]\frac{\partial\tilde{\rho}_{\epsilon,k}}{\partial k}. \tag{S3}$$

Under the assumption $c(\rho) = \xi\rho^2$, taking the derivative of both sides of (8) with the identity $\tilde{\rho} = \tilde{\rho}_{\epsilon,k}$ with respect to $k$ leads to

$$(\gamma^2\rho_v + \gamma^2\rho_\zeta + \rho_x\rho_\zeta^2)\frac{\partial\tilde{\rho}_{\epsilon,k}}{\partial k} + (\gamma^2 + 2\rho_x\rho_\zeta)\left(\tilde{\rho}_{\epsilon,k}^2 + 2k\tilde{\rho}_{\epsilon,k}\frac{\partial\tilde{\rho}_{\epsilon,k}}{\partial k}\right) + 2\rho_x k\tilde{\rho}_{\epsilon,k}^3 + 3\rho_x k^2\tilde{\rho}_{\epsilon,k}^2\frac{\partial\tilde{\rho}_{\epsilon,k}}{\partial k} = 0,$$

from which we have

$$\frac{\partial\tilde{\rho}_{\epsilon,k}}{\partial k} = -\frac{(\gamma^2 + 2\rho_x\rho_\zeta)\tilde{\rho}_{\epsilon,k}^2 + 2\rho_x k\tilde{\rho}_{\epsilon,k}^3}{\gamma^2\rho_v + \gamma^2\rho_\zeta + \rho_x\rho_\zeta^2 + 2(\gamma^2 + 2\rho_x\rho_\zeta)k\tilde{\rho}_{\epsilon,k} + 3\rho_x k^2\tilde{\rho}_{\epsilon,k}^2}$$
$$\stackrel{(9)}{=} -\frac{(\gamma^2 + 2\rho_x\tilde{z}_k)\tilde{\rho}_{\epsilon,k}^2}{\frac{\gamma}{4\xi\tilde{\rho}_{\epsilon,k}} + \gamma^2 k\tilde{\rho}_{\epsilon,k} + 2\rho_x\tilde{z}_k k\tilde{\rho}_{\epsilon,k}}. \tag{S4}$$

Substituting (S4) into (S3) and noting $c(\rho) = \xi\rho^2$ again, we obtain

$$\begin{aligned}\frac{\partial \widetilde{CE}(k)}{\partial k} &\propto 4\xi\gamma^6\rho_x^{-1}\tilde{\rho}_{\epsilon,k}^2 - 8\xi\gamma^4\rho_v\tilde{\rho}_{\epsilon,k}^2 + 4\xi\gamma^4\tilde{z}_k\tilde{\rho}_{\epsilon,k}^2 + 2\gamma\rho_x\tilde{z}_k\tilde{\rho}_{\epsilon,k} + 3\gamma^3\tilde{\rho}_{\epsilon,k} \\ &\quad - 2\gamma^3\tilde{z}_k - \gamma\rho_x\tilde{z}_k^2 + \gamma^3\rho_v - \gamma^5\rho_x^{-1} \\ &=: \tilde{A}(k).\end{aligned}$$

Next, we show that $\partial\tilde{A}(k)/\partial k < 0$ hold for any $k \in [1, N]$. Taking the derivative of $\tilde{A}(k)$ with respect to $k$, we have

$$\begin{aligned}\frac{\partial \tilde{A}(k)}{\partial k} &= \Big[8\xi\gamma^6\rho_x^{-1}\tilde{\rho}_{\epsilon,k} - 16\xi\gamma^4\rho_v\tilde{\rho}_{\epsilon,k} + 4\xi\gamma^4\tilde{\rho}_{\epsilon,k}^2 k + 8\xi\gamma^4\tilde{z}_k\tilde{\rho}_{\epsilon,k} + 2\gamma\rho_x k\tilde{\rho}_{\epsilon,k} + 2\gamma\rho_x\tilde{z}_k \\ &\quad + 3\gamma^3 - 2\gamma^3 k - 2\gamma\rho_x k\tilde{z}_k\Big]\frac{\partial\tilde{\rho}_{\epsilon,k}}{\partial k} + 4\xi\gamma^4\tilde{\rho}_{\epsilon,k}^3 + 2\gamma\rho_x\tilde{\rho}_{\epsilon,k}^2 - 2\gamma^3\tilde{\rho}_{\epsilon,k} - 2\gamma\rho_x\tilde{z}_k\tilde{\rho}_{\epsilon,k} \\ &\overset{(S4)}{=} -\frac{1}{\frac{\gamma}{4\xi\tilde{\rho}_{\epsilon,k}} + \gamma^2 k\tilde{\rho}_{\epsilon,k} + 2\rho_x\tilde{z}_k k\tilde{\rho}_{\epsilon,k}}\Big[8\xi\gamma^8\rho_x^{-1}\tilde{\rho}_{\epsilon,k}^3 - 16\xi\gamma^6\rho_v\tilde{\rho}_{\epsilon,k}^3 + 4\xi\gamma^6 k\tilde{\rho}_{\epsilon,k}^4 \\ &\quad + 8\xi\gamma^6\tilde{z}_k\tilde{\rho}_{\epsilon,k}^3 + 2\gamma^3\rho_x k\tilde{\rho}_{\epsilon,k}^3 + 2\gamma^3\rho_x\tilde{z}_k\tilde{\rho}_{\epsilon,k}^2 + 3\gamma^5\tilde{\rho}_{\epsilon,k}^2 - 2\gamma^5 k\tilde{\rho}_{\epsilon,k}^2 \\ &\quad - 2\gamma^3\rho_x\tilde{z}_k k\tilde{\rho}_{\epsilon,k}^2 + 16\xi\gamma^6\tilde{z}_k\tilde{\rho}_{\epsilon,k}^3 - 32\xi\gamma^4\rho_v\rho_x\tilde{z}_k\tilde{\rho}_{\epsilon,k}^3 + 8\xi\gamma^4\rho_x\tilde{z}_k k\tilde{\rho}_{\epsilon,k}^4 \\ &\quad + 16\xi\gamma^4\rho_x\tilde{z}_k^2\tilde{\rho}_{\epsilon,k}^3 + 4\gamma\rho_x^2\tilde{z}_k k\tilde{\rho}_{\epsilon,k}^3 + 4\gamma\rho_x^2\tilde{z}_k^2\tilde{\rho}_{\epsilon,k}^2 + 6\gamma^3\rho_x\tilde{z}_k\tilde{\rho}_{\epsilon,k}^2 - 4\gamma^3\rho_x\tilde{z}_k k\tilde{\rho}_{\epsilon,k}^2 \\ &\quad - 4\gamma\rho_x^2\tilde{z}_k^2 k\tilde{\rho}_{\epsilon,k}^2 - \gamma^5\tilde{\rho}_{\epsilon,k}^2 - \frac{1}{2\xi}\gamma^2\rho_x\tilde{\rho}_{\epsilon,k} + \frac{1}{2\xi}\gamma^4 + \frac{1}{2\xi}\gamma^2\rho_x\tilde{z}_k - 4\xi\gamma^6 k\tilde{\rho}_{\epsilon,k}^4 \\ &\quad - 2\gamma^3\rho_x k\tilde{\rho}_{\epsilon,k}^3 + 2\gamma^5 k\tilde{\rho}_{\epsilon,k}^2 + 2\gamma^3\rho_x\tilde{z}_k k\tilde{\rho}_{\epsilon,k}^2 - 8\xi\gamma^4\rho_x\tilde{z}_k k\tilde{\rho}_{\epsilon,k}^4 - 4\gamma\rho_x^2\tilde{z}_k k\tilde{\rho}_{\epsilon,k}^3 \\ &\quad + 4\gamma^3\rho_x\tilde{z}_k k\tilde{\rho}_{\epsilon,k}^2 + 4\gamma\rho_x^2\tilde{z}_k^2 k\tilde{\rho}_{\epsilon,k}^2\Big] \\ &\propto -\Big[8\xi\gamma^8\rho_x^{-1}\tilde{\rho}_{\epsilon,k}^3 - 16\xi\gamma^6\rho_v\tilde{\rho}_{\epsilon,k}^3 + 8\xi\gamma^6\tilde{z}_k\tilde{\rho}_{\epsilon,k}^3 + 8\gamma^3\rho_x\tilde{z}_k\tilde{\rho}_{\epsilon,k}^2 + 2\gamma^5\tilde{\rho}_{\epsilon,k}^2 \\ &\quad + 16\xi\gamma^6\tilde{z}_k\tilde{\rho}_{\epsilon,k}^3 - 32\xi\gamma^4\rho_v\rho_x\tilde{z}_k\tilde{\rho}_{\epsilon,k}^3 + 4\gamma\rho_x^2\tilde{z}_k k\tilde{\rho}_{\epsilon,k}^3 + 4\gamma\rho_x^2\tilde{z}_k\tilde{\rho}_{\epsilon,k}^2(\tilde{z}_k - k\tilde{\rho}_{\epsilon,k}) \\ &\quad + \frac{1}{2\xi}\gamma^2\rho_x(\tilde{z}_k - \tilde{\rho}_{\epsilon,k}) + 16\xi\gamma^4\rho_x\tilde{z}_k^2\tilde{\rho}_{\epsilon,k}^3 + \frac{1}{2\xi}\gamma^4\Big] \\ &=: -\tilde{B}(k). \qquad \text{(S5)}\end{aligned}$$

Using (8), we have

$$-4\xi\gamma^2\rho_v\tilde{\rho}_{\epsilon,k} = 4\xi\gamma^2\tilde{z}_k\tilde{\rho}_{\epsilon,k} + 4\xi\rho_x\tilde{z}_k^2\tilde{\rho}_{\epsilon,k} - \gamma.$$

Thus,

$$\begin{aligned} -16\xi\gamma^6\rho_v\tilde{\rho}_{\epsilon,k}^3 &= 4\gamma^4\tilde{\rho}_{\epsilon,k}^2\left[4\xi\gamma^2\tilde{z}_k\tilde{\rho}_{\epsilon,k} + 4\xi\rho_x\tilde{z}_k^2\tilde{\rho}_{\epsilon,k} - \gamma\right] \\ &= 16\xi\gamma^6\tilde{z}_k\tilde{\rho}_{\epsilon,k}^3 + 16\xi\gamma^4\rho_x\tilde{z}_k^2\tilde{\rho}_{\epsilon,k}^3 - 4\gamma^5\tilde{\rho}_{\epsilon,k}^2, \end{aligned} \tag{S6}$$

and

$$\begin{aligned} -32\xi\gamma^4\rho_v\rho_x\tilde{z}_k\tilde{\rho}_{\epsilon,k}^3 &= 8\gamma^2\rho_x\tilde{z}_k\tilde{\rho}_{\epsilon,k}^2\left[4\xi\gamma^2\tilde{z}_k\tilde{\rho}_{\epsilon,k} + 4\xi\rho_x\tilde{z}_k^2\tilde{\rho}_{\epsilon,k} - \gamma\right] \\ &= 32\xi\gamma^4\rho_x\tilde{z}_k^2\tilde{\rho}_{\epsilon,k}^3 + 32\xi\gamma^2\rho_x^2\tilde{z}_k^3\tilde{\rho}_{\epsilon,k}^3 - 8\gamma^3\rho_x\tilde{z}_k\tilde{\rho}_{\epsilon,k}^2. \end{aligned} \tag{S7}$$

Substituting (S6) and (S7) into (S5), and noting that $\tilde{z}_k > k\tilde{\rho}_{\epsilon,k} \geq \tilde{\rho}_{\epsilon,k}$, we can see that

$$\begin{aligned} \tilde{B}(k) &= 8\xi\gamma^8\rho_x^{-1}\tilde{\rho}_{\epsilon,k}^3 + 24\xi\gamma^6\tilde{z}_k\tilde{\rho}_{\epsilon,k}^3 + 32\xi\gamma^2\rho_x^2\tilde{z}_k^3\tilde{\rho}_{\epsilon,k}^3 + 16\xi\gamma^6\tilde{z}_k\tilde{\rho}_{\epsilon,k}^3 + 4\gamma\rho_x^2\tilde{z}_k k\tilde{\rho}_{\epsilon,k}^3 \\ &\quad + 4\gamma\rho_x^2\tilde{z}_k\tilde{\rho}_{\epsilon,k}^2(\tilde{z}_k - k\tilde{\rho}_{\epsilon,k}) + \frac{1}{2\xi}\gamma^2\rho_x(\tilde{z}_k - \tilde{\rho}_{\epsilon,k}) + 64\xi\gamma^4\rho_x\tilde{z}_k^2\tilde{\rho}_{\epsilon,k}^3 + \frac{1}{2\xi}\gamma^4 - 2\gamma^5\tilde{\rho}_{\epsilon,k}^2 \\ &\overset{(8)}{=} 8\xi\gamma^8\rho_x^{-1}\tilde{\rho}_{\epsilon,k}^3 + 40\xi\gamma^6\tilde{z}_k\tilde{\rho}_{\epsilon,k}^3 + 32\xi\gamma^2\rho_x^2\tilde{z}_k^3\tilde{\rho}_{\epsilon,k}^3 + 4\gamma\rho_x^2\tilde{z}_k k\tilde{\rho}_{\epsilon,k}^3 \\ &\quad + 4\gamma\rho_x^2\tilde{z}_k\tilde{\rho}_{\epsilon,k}^2(\tilde{z}_k - k\tilde{\rho}_{\epsilon,k}) + \frac{1}{2\xi}\gamma^2\rho_x(\tilde{z}_k - \tilde{\rho}_{\epsilon,k}) + 64\xi\gamma^4\rho_x\tilde{z}_k^2\tilde{\rho}_{\epsilon,k}^3 \\ &\quad + 2\gamma^5\rho_v\tilde{\rho}_{\epsilon,k} + 2\gamma^5\tilde{\rho}_{\epsilon,k}(\tilde{z}_k - \tilde{\rho}_{\epsilon,k}) + 2\gamma^3\rho_x\tilde{z}_k^2\tilde{\rho}_{\epsilon,k} \\ &> 0, \end{aligned}$$

which yields $\partial\tilde{A}(k)/\partial k < 0$. This completes the proof. □

## S2 The Welfare Effects in Subsection 4.3

Here we provide the detailed proofs of the conclusions on the welfare effects in Subsection 4.3. According to (A.1), the certainty equivalent is given as follows

$$\begin{aligned} \mathbb{E}\left[U(\hat{d}_{i,g}^*(v-\hat{p}) - c(\hat{\rho}_{\epsilon,k}))\right] &= -e^{\gamma c(\hat{\rho}_{\epsilon,k})}\sqrt{\frac{1}{\rho_v + \hat{\rho}_\theta + \hat{\rho}_{\epsilon,k} + (k-1)\rho_y}\frac{1}{(1-\hat{\alpha}_v)^2\rho_v^{-1} + \hat{\alpha}_x^2\rho_x^{-1}}} \\ &= -e^{\gamma c(\hat{\rho}_{\epsilon,k})}\sqrt{1 - \frac{\gamma^4\rho_x^{-1} + \gamma^2\hat{z}_k}{\rho_x\hat{z}_k^2 + 2\gamma^2\hat{z}_k + \gamma^2\rho_v + \gamma^4\rho_x^{-1}}}, \end{aligned} \tag{S8}$$

where $\hat{d}^*_{i,g}$ refers to the optimal demand of trader $i$ in group $g$ when information is shared imperfectly, the first equality follows from (10) and the projection theorem for normal random variables, and (S8) follows from the expressions of $\hat{\alpha}_v$, $\hat{\alpha}_x$ and $\hat{\rho}_\theta$ defined by (11), (12) and (13). Thus, according to (S8), we can see that the certainty equivalent with imperfect information sharing is given by

$$\begin{aligned}\widehat{CE}(k) &:= -\frac{1}{\gamma}\log\left[\mathbb{E}(\exp\{-\gamma\hat{d}^*_{i,g}(v-p)\})\right] - c(\hat{\rho}_{\epsilon,k}) \\ &= \frac{1}{2\gamma}\log\left[\frac{\rho_x\hat{z}_k^2 + 2\gamma^2\hat{z}_k + \gamma^2\rho_v + \gamma^4\rho_x^{-1}}{\rho_x\hat{z}_k^2 + \gamma^2\hat{z}_k + \gamma^2\rho_v}\right] - c(\hat{\rho}_{\epsilon,k}).\end{aligned} \tag{S9}$$

We then show that $\partial\widehat{CE}(k)/\partial k \propto \hat{A}(k)$ and $\hat{A}(k)$ is strictly decreasing in $k$. First, according to (14), rewriting (S9) as

$$\begin{aligned}\widehat{CE}(k) &= \left[\frac{1}{2\gamma}\log\left(\rho_x\hat{z}_k^2 + 2\gamma^2\hat{z}_k + \gamma^2\rho_v + \gamma^4\rho_x^{-1}\right)\right] - \left[\frac{1}{2\gamma}\log\left(\frac{\gamma}{2c'(\hat{\rho}_{\epsilon,k})}\right) + c(\hat{\rho}_{\epsilon,k})\right] \\ &=: \hat{F}_1(k) - \hat{F}_2(k).\end{aligned}$$

Then,

$$\begin{aligned}\frac{\partial\widehat{CE}(k)}{\partial k} &= \frac{\partial\hat{F}_1(k)}{\partial\hat{z}_k}\frac{\partial\hat{z}_k}{\partial k} - \frac{\partial\hat{F}_2(k)}{\partial\hat{\rho}_{\epsilon,k}}\frac{\partial\hat{\rho}_{\epsilon,k}}{\partial k} \\ &= \frac{1}{2\gamma}\frac{2\rho_x\hat{z}_k + 2\gamma^2}{\rho_x\hat{z}_k^2 + 2\gamma^2\hat{z}_k + \gamma^2\rho_v + \gamma^4\rho_x^{-1}}\frac{\partial\hat{z}_k}{\partial k} - \left[-\frac{c''(\hat{\rho}_{\epsilon,k})}{2\gamma c'(\hat{\rho}_{\epsilon,k})} + c'(\hat{\rho}_{\epsilon,k})\right]\frac{\partial\hat{\rho}_{\epsilon,k}}{\partial k} \\ &\overset{(15)}{=} \frac{1}{2\gamma}\frac{2\rho_x\hat{z}_k + 2\gamma^2}{\rho_x\hat{z}_k^2 + 2\gamma^2\hat{z}_k + \gamma^2\rho_v + \gamma^4\rho_x^{-1}}\left[\left(1 + \frac{(k-1)\rho_\zeta^2}{(\rho_\zeta + \hat{\rho}_{\epsilon,k})^2}\right)\frac{\partial\hat{\rho}_{\epsilon,k}}{\partial k} + \frac{\rho_\zeta\hat{\rho}_{\epsilon,k}}{\rho_\zeta + \hat{\rho}_{\epsilon,k}}\right] \\ &\quad - \left[-\frac{c''(\hat{\rho}_{\epsilon,k})}{2\gamma c'(\hat{\rho}_{\epsilon,k})} + c'(\hat{\rho}_{\epsilon,k})\right]\frac{\partial\hat{\rho}_{\epsilon,k}}{\partial k}.\end{aligned} \tag{S10}$$

With the assumption $c(\rho) = \xi\rho^2$, taking the partial derivative of both sides of (14) with the identity $\hat{\rho} = \hat{\rho}_{\epsilon,k}$ with respect to $k$ leads to

$$(\gamma^2\rho_v + \gamma^2\hat{z}_k + \rho_x\hat{z}_k^2)\frac{\partial\hat{\rho}_{\epsilon,k}}{\partial k} + (\gamma^2\hat{\rho}_{\epsilon,k} + 2\rho_x\hat{\rho}_{\epsilon,k}\hat{z}_k)\left[\left(1 + \frac{(k-1)\rho_\zeta^2}{(\rho_\zeta + \hat{\rho}_{\epsilon,k})^2}\right)\frac{\partial\hat{\rho}_{\epsilon,k}}{\partial k} + \frac{\rho_\zeta\hat{\rho}_{\epsilon,k}}{\rho_\zeta + \hat{\rho}_{\epsilon,k}}\right] = 0,$$

from which we can see that

$$\frac{\partial \hat{\rho}_{\epsilon,k}}{\partial k} = -\frac{\frac{\rho_\zeta \hat{\rho}_{\epsilon,k}^2(\gamma^2+2\rho_x\hat{z}_k)}{\rho_\zeta+\hat{\rho}_{\epsilon,k}}}{\gamma^2\rho_v + \gamma^2\hat{z}_k + \rho_x\hat{z}_k^2 + \frac{(k\rho_\zeta^2+2\rho_\zeta\hat{\rho}_{\epsilon,k}+\hat{\rho}_{\epsilon,k}^2)(\gamma^2+2\rho_x\hat{z}_k)\hat{\rho}_{\epsilon,k}}{(\rho_\zeta+\hat{\rho}_{\epsilon,k})^2}}$$
$$\overset{(15)}{=} -\frac{\frac{\rho_\zeta \hat{\rho}_{\epsilon,k}^2(\gamma^2+2\rho_x\hat{z}_k)}{\rho_\zeta+\hat{\rho}_{\epsilon,k}}}{\frac{\gamma}{4\xi\hat{\rho}_{\epsilon,k}} + \frac{(k\rho_\zeta^2+2\rho_\zeta\hat{\rho}_{\epsilon,k}+\hat{\rho}_{\epsilon,k}^2)(\gamma^2+2\rho_x\hat{z}_k)\hat{\rho}_{\epsilon,k}}{(\rho_\zeta+\hat{\rho}_{\epsilon,k})^2}}. \tag{S11}$$

Substituting (S11) into (S10) and reminding that $c(\rho) = \xi\rho^2$ again, we can obtain

$$\frac{\partial \widehat{CE}(k)}{\partial k} \propto 12\xi\gamma^4\hat{z}_k\hat{\rho}_{\epsilon,k}^2 + 8\xi\gamma^2\rho_x\hat{z}_k^2\hat{\rho}_{\epsilon,k}^2 + 4\xi\gamma^6\rho_x^{-1}\hat{\rho}_{\epsilon,k}^2 + 2\gamma\rho_x\hat{z}_k\hat{\rho}_{\epsilon,k} + \gamma^3\hat{\rho}_{\epsilon,k} - \gamma\rho_x\hat{z}_k^2$$
$$- 2\gamma^3\hat{z}_k - \gamma^5\rho_x^{-1} + \gamma^3\rho_v$$
$$=: \hat{A}(k).$$

We next show that $\partial\hat{A}(k)/\partial k < 0$ hold for any $k \in [1, N]$. Taking derivative of $\hat{A}(k)$ with respect to $k$, we can see that

$$\frac{\partial \hat{A}(k)}{\partial k} = 12\xi\gamma^4\hat{\rho}_{\epsilon,k}^2\frac{\partial \hat{z}_k}{\partial k} + 24\xi\gamma^4\hat{z}_k\hat{\rho}_{\epsilon,k}\frac{\partial \hat{\rho}_{\epsilon,k}}{\partial k} + 16\xi\gamma^2\rho_x\hat{z}_k\hat{\rho}_{\epsilon,k}^2\frac{\partial \hat{z}_k}{\partial k} + 16\xi\gamma^2\rho_x\hat{z}_k^2\hat{\rho}_{\epsilon,k}\frac{\partial \hat{\rho}_{\epsilon,k}}{\partial k}$$
$$+ 8\xi\gamma^6\rho_x^{-1}\hat{\rho}_{\epsilon,k}\frac{\partial \hat{\rho}_{\epsilon,k}}{\partial k} + 2\gamma\rho_x\hat{\rho}_{\epsilon,k}\frac{\partial \hat{z}_k}{\partial k} + 2\gamma\rho_x\hat{z}_k\frac{\hat{\rho}_{\epsilon,k}}{\partial k} + \gamma^3\frac{\partial \hat{\rho}_{\epsilon,k}}{\partial k} - 2\gamma\rho_x\hat{z}_k\frac{\partial \hat{z}_k}{\partial k} - 2\gamma^3\frac{\partial \hat{z}_k}{\partial k}$$
$$= \Bigg[\frac{(\rho_\zeta+\hat{\rho}_{\epsilon,k})^2 + (k-1)\rho_\zeta^2}{(\rho_\zeta+\hat{\rho}_{\epsilon,k})^2}\left(12\xi\gamma^4\hat{\rho}_{\epsilon,k}^2 + 16\xi\gamma^2\rho_x\hat{z}_k\hat{\rho}_{\epsilon,k}^2 + 2\gamma\rho_x\hat{\rho}_{\epsilon,k} - 2\gamma\rho_x\hat{z}_k - 2\gamma^3\right)$$
$$+ 24\xi\gamma^4\hat{z}_k\hat{\rho}_{\epsilon,k} + 16\xi\gamma^2\rho_x\hat{z}_k^2\hat{\rho}_{\epsilon,k} + 8\xi\gamma^6\rho_x^{-1}\hat{\rho}_{\epsilon,k} + 2\gamma\rho_x\hat{z}_k + \gamma^3\Bigg]\frac{\partial \hat{\rho}_{\epsilon,k}}{\partial k}$$
$$+ \left(12\xi\gamma^4\hat{\rho}_{\epsilon,k}^2 + 16\xi\gamma^2\rho_x\hat{z}_k\hat{\rho}_{\epsilon,k}^2 + 2\gamma\rho_x\hat{\rho}_{\epsilon,k} - 2\gamma\rho_x\hat{z}_k - 2\gamma^3\right)\frac{\rho_\zeta\hat{\rho}_{\epsilon,k}}{\rho_\zeta+\hat{\rho}_{\epsilon,k}}$$
$$= -\frac{1}{\frac{\gamma}{4\xi\hat{\rho}_{\epsilon,k}} + \frac{(k\rho_\zeta^2+2\rho_\zeta\hat{\rho}_{\epsilon,k}+\hat{\rho}_{\epsilon,k}^2)(\gamma^2+2\rho_x\hat{z}_k)\hat{\rho}_{\epsilon,k}}{(\rho_\zeta+\hat{\rho}_{\epsilon,k})^2}}\Bigg[\Bigg(\frac{(\rho_\zeta+\hat{\rho}_{\epsilon,k})^2 + (k-1)\rho_\zeta^2}{(\rho_\zeta+\hat{\rho}_{\epsilon,k})^2}$$
$$\times\left(12\xi\gamma^4\hat{\rho}_{\epsilon,k}^2 + 16\xi\gamma^2\rho_x\hat{z}_k\hat{\rho}_{\epsilon,k}^2 + 2\gamma\rho_x\hat{\rho}_{\epsilon,k} - 2\gamma\rho_x\hat{z}_k - 2\gamma^3\right)$$
$$+ 24\xi\gamma^4\hat{z}_k\hat{\rho}_{\epsilon,k} + 16\xi\gamma^2\rho_x\hat{z}_k^2\hat{\rho}_{\epsilon,k} + 8\xi\gamma^6\rho_x^{-1}\hat{\rho}_{\epsilon,k} + 2\gamma\rho_x\hat{z}_k + \gamma^3\Bigg)\frac{\rho_\zeta\hat{\rho}_{\epsilon,k}^2(\gamma^2+2\rho_x\hat{z}_k)}{\rho_\zeta+\hat{\rho}_{\epsilon,k}}$$
$$- \left(12\xi\gamma^4\hat{\rho}_{\epsilon,k}^2 + 16\xi\gamma^2\rho_x\hat{z}_k\hat{\rho}_{\epsilon,k}^2 + 2\gamma\rho_x\hat{\rho}_{\epsilon,k} - 2\gamma\rho_x\hat{z}_k - 2\gamma^3\right)$$

$$
\begin{aligned}
&\quad \times \frac{\rho_\zeta \hat{\rho}_{\epsilon,k}}{\rho_\zeta + \hat{\rho}_{\epsilon,k}} \left( \frac{\gamma}{4\xi \hat{\rho}_{\epsilon,k}} + \frac{(k\rho_\zeta^2 + 2\rho_\zeta \hat{\rho}_{\epsilon,k} + \hat{\rho}_{\epsilon,k}^2)(\gamma^2 + 2\rho_x \hat{z}_k)\hat{\rho}_{\epsilon,k}}{(\rho_\zeta + \hat{\rho}_{\epsilon,k})^2} \right) \Bigg] \\
&\propto - \Big[ \left(24\xi\gamma^4 \hat{z}_k \hat{\rho}_{\epsilon,k} + 16\xi\gamma^2 \rho_x \hat{z}_k^2 \hat{\rho}_{\epsilon,k} + 8\xi\gamma^6 \rho_x^{-1} \hat{\rho}_{\epsilon,k} + 2\gamma\rho_x \hat{z}_k + \gamma^3\right) \left(\gamma^2 + 2\rho_x \hat{z}_k\right) \hat{\rho}_{\epsilon,k} \\
&\quad - \left(12\xi\gamma^4 \hat{\rho}_{\epsilon,k}^2 + 16\xi\gamma^2 \rho_x \hat{z}_k \hat{\rho}_{\epsilon,k}^2 + 2\gamma\rho_x \hat{\rho}_{\epsilon,k} - 2\gamma\rho_x \hat{z}_k - 2\gamma^3\right) \frac{\gamma}{4\xi \hat{\rho}_{\epsilon,k}} \Big] \\
&= - \Big[ 24\xi\gamma^6 \hat{z}_k \hat{\rho}_{\epsilon,k}^2 + 16\xi\gamma^4 \rho_x \hat{z}_k^2 \hat{\rho}_{\epsilon,k}^2 + 8\xi\gamma^8 \rho_x^{-1} \hat{\rho}_{\epsilon,k}^2 + 2\gamma^3 \rho_x \hat{z}_k \hat{\rho}_{\epsilon,k} + \gamma^5 \hat{\rho}_{\epsilon,k} \\
&\quad + 48\xi\gamma^4 \rho_x \hat{z}_k^2 \hat{\rho}_{\epsilon,k}^2 + 32\xi\gamma^2 \rho_x^2 \hat{z}_k^3 \hat{\rho}_{\epsilon,k}^2 + 16\xi\gamma^6 \hat{z}_k \hat{\rho}_{\epsilon,k}^2 + 4\gamma\rho_x^2 \hat{z}_k^2 \hat{\rho}_{\epsilon,k} + 2\gamma^3 \rho_x \hat{z}_k \hat{\rho}_{\epsilon,k} \\
&\quad - 3\gamma^5 \hat{\rho}_{\epsilon,k} - 4\gamma^3 \rho_x \hat{z}_k \hat{\rho}_{\epsilon,k} - \frac{\gamma^2 \rho_x}{2\xi} + \frac{\gamma^2 \rho_x \hat{z}_k}{2\xi \hat{\rho}_{\epsilon,k}} + \frac{\gamma^4}{2\xi \hat{\rho}_{\epsilon,k}} \Big] \\
&\overset{(14)}{=} - \Big[ 24\xi\gamma^6 \hat{z}_k \hat{\rho}_{\epsilon,k}^2 + 16\xi\gamma^4 \rho_x \hat{z}_k^2 \hat{\rho}_{\epsilon,k}^2 + 8\xi\gamma^8 \rho_x^{-1} \hat{\rho}_{\epsilon,k}^2 + 48\xi\gamma^4 \rho_x \hat{z}_k^2 \hat{\rho}_{\epsilon,k}^2 \\
&\quad + 32\xi\gamma^2 \rho_x^2 \hat{z}_k^3 \hat{\rho}_{\epsilon,k}^2 + 16\xi\gamma^6 \hat{z}_k \hat{\rho}_{\epsilon,k}^2 + 4\gamma\rho_x^2 \hat{z}_k^2 \hat{\rho}_{\epsilon,k} + \frac{\gamma^2 \rho_x}{2\xi} \left( \frac{\hat{z}_k}{\hat{\rho}_{\epsilon,k}} - 1 \right) \\
&\quad + 2\gamma^5 \rho_v + 2\gamma^5 (\hat{z}_k - \hat{\rho}_{\epsilon,k}) + 2\gamma^3 \rho_x \hat{z}_k^2 \Big] \\
&< 0,
\end{aligned}
$$

where the inequality follows from (15). This completes the proof. □