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Brief paper

Target containment control of multi-agent systems with random switching interconnection topologies[☆]

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ABSTRACT

In this paper, the distributed containment control is considered for a second-order multi-agent system guided by multiple leaders with random switching topologies. The multi-leader control problem is investigated via a combination of convex analysis and stochastic process. The interaction topology between agents is described by a continuous-time irreducible Markov chain. A necessary and sufficient condition is obtained to make all the mobile agents almost surely asymptotically converge to the static convex leader set. Moreover, conditions on the tracking estimation are provided for the convex target set determined by moving multiple leaders.

1. Introduction

Recent years have witnessed a huge and rapidly growing literature concerned with multi-agent problems due to the broad applications in various disciplines. The leader–follower coordination, as one of the important problems of multi-agent networks, has been studied in the last decade, with significant results obtained for first-order or second-order multi-agent systems. Static-leader cases were studied with jointly-connected interaction topologies in Jadbabaie, Lin, and Morse (2003). Moreover, potential function approaches were used to drive the agents to follow a desired trajectory in Olfati-Saber (2006) and similar results under relaxed assumptions were obtained in Ren and Beard (2008) and Su, Wang, and Lin (2009). To follow a moving leader with unmeasurable velocity, distributed observers were designed for second-order multi-agent systems in Hong, Chen, and Bushnell (2008). Also, an estimator-based tracking problem was investigated for a leader–follower system with measurement noises in Hu and Feng (2010).

Due to the practical demand, multi-agent coordination with multiple leaders becomes more and more important since multiple leaders may be useful to achieve effectively the containment or guidance of an agent group in a target region (see Couzin, Krause, Franks, and Levin (2005)). Target aggregation or containment with multiple leaders was developed, aiming at containing a group of agents in a specific target region. Containment control schemes were proposed to make the agents stay in the convex set spanned by the multiple leaders in Ji, Ferrari-Trecate, Egerstedt, and Buffa (2008). The target containment of nonlinear multi-agent systems with different switching topologies was considered to contain a group of agents guided by leaders in a given target set in Shi and Hong (2009). Also, a distributed control method was reported for multi-agent containment in Cao and Ren (2010). Additionally, the attitude containment control was studied in Dimarogonas, Tsiotras, and Kyriakopoulos (2009), while finite-time control law was designed for containment in Meng, Ren, and You (2010).

Random switching topologies were also investigated for multi-agent coordination algorithms due to many practical backgrounds including gossip algorithms and communication patterns (for example, Boyd, Ghosh, Prabhakar, and Shah (2006) and Matei, Martins, and Baras (2009)). In fact, during the information transmission, packet drop and node failure phenomena can be described as random switching graph processes, and multi-agent consensus with various random graph processes was also important. To solve the related coordination problems, different approaches were proposed. For example, the asymptotic almost sure consensus is achieved over random information networks in Porfiri and Stilwell (2007), where the existence of any edge in

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a topology is probabilistic and independent from the existence of any other edge. Moreover, similar results were obtained for stationary and ergodic graph processes in Tahbaz-Salehi and Jadbabaie (2010), while the mean square consensus problem was discussed for a second-order discrete-time system with Markovian graphs in Zhang and Tian (2009). Additionally, Liu, Lu, and Chen (2011) also investigated consensus problem based on adapted stochastic processes.

To our knowledge, there is no theoretical result on containment of second-order multi-agent systems with random switching interconnections. The objective of the paper is to study the containment control for a second-order multi-agent system with a target set specified by multiple leaders. Here, we develop a new method to solve the problem with the help of both convex analysis and stochastic process analysis, because the existing methods on random consensus used in Porfiri and Stilwell (2007), Tahbaz-Salehi and Jadbabaie (2010) and Zhang and Tian (2009), or the containment methods for deterministic systems proposed in Cao and Ren (2010) and Ji et al. (2008), cannot be applied to solve our problem; we solve the containment of the second-order agent systems with switching topologies, which is more complicated than the first-order agent model with deterministic switching studied in Shi and Hong (2009). Additionally, we investigate set containment for continuous-time systems, different from many existing random consensus results for discrete-time systems.

Notation. I_n is the $n \times n$ identity matrix; For a given vector x, x^T stands for its transpose, $\|x\|_2$ for its Euclidean norm; For a given matrix F, $\|F\|_\infty$ stands for its infinite norm, $\exp(F)$ for its matrix exponential, $(F)_{ij}$ for its i-th row and j-th column entry; $(W)_{**}$ denotes the $2n \times 2n$ left upper block of matrix $W \in R^{(2n+l)\times(2n+l)}$; $\otimes$ denotes Kronecker product.

2. Preliminaries and formulation

In this section, we introduce preliminary knowledge about graph theory and stochastic process, and then our problem formulation.

It is known that the interaction topology of a multi-agent system consisting of n agents (followers) and l leaders can be described by a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the set of nodes $\mathcal{V} = \mathcal{I} \bigcup \mathcal{L}$ and the set of arcs $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$. Without loss of generality, we assume the first n agents as the followers and the last l agents as the leaders. Let $\mathcal{I} = \{1, \ldots, n\}$ and $\mathcal{L} = \{n+1, \ldots, n+l\}$ denote the index sets of followers and leaders, respectively. $(i, j) \in \mathcal{E}$ means that there is an arc from node i to node j (or equivalently, node j is a neighbor of node i). The adjacency matrix associated with the graph is denoted as $A = [a_{ij}]_{(n+l)\times(n+l)}$ with nonnegative adjacency elements a_{ij}. The element a_{ij} of matrix A associated with arc (i, j) is positive, i.e., $a_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$. There is no self-loop in $\mathcal{G}$, i.e., $a_{ii} = 0$ for all $i \in \mathcal{V}$. In our problem, $a_{ij} = 0$ for all $i \in \mathcal{L}$ and $j \in \mathcal{V}$. A path from i to j in $\mathcal{G}$ is a sequence $i_0, i_1, \ldots, i_\iota$ of distinct nodes such that $(i_{k-1}, i_k) \in \mathcal{E}$ for $k = 1, \ldots, \iota$, where $i_0 = i, i_\iota = j$. Node j is reachable from node i if there is at least one path from i to j. Leader set $\mathcal{L}$ is reachable from node i if there exists at least one leader $j \in \mathcal{L}$ such that j is reachable from i. Moreover, $\mathcal{L}$ is globally reachable in $\mathcal{G}$ if it is reachable from every node of $\mathcal{I}$.

Given digraph $\mathcal{G}$, $\mathcal{E}(\mathcal{G})$ and $A(\mathcal{G})$ denote the set of arcs and the adjacency matrix of $\mathcal{G}$, respectively. The set of neighbors of node i in $\mathcal{I}$ and $\mathcal{L}$ are denoted by $\mathcal{N}_{if}(\mathcal{G}) = \{j|(i, j) \in \mathcal{E}(\mathcal{G}), j \in \mathcal{I}\}$, $\mathcal{N}_{il}(\mathcal{G}) = \{j|(i, j) \in \mathcal{E}(\mathcal{G}), j \in \mathcal{L}\}$, respectively. $\bigcup_{1\le r\le p} \mathcal{G}_r$ denotes the union graph with nodes set $\mathcal{V}$ and arcs set $\bigcup_{1\le r\le p} \mathcal{E}(\mathcal{G}_r)$. Let $\mathcal{G}^f$ be the induced subgraph of $\mathcal{G}$ with all followers as nodes. The degree matrix of $\mathcal{G}^f$ is a diagonal matrix $D^f = \mathrm{diag}\{d_1^f, \ldots, d_n^f\}$ with $d_i^f = \sum_{1\le j\le n} a_{ij} (1 \le i \le n)$ and the Laplacian matrix of $\mathcal{G}_f$ is defined as $L^f = D^f - A^f$, where A^f is the adjacency matrix of $\mathcal{G}^f$ (referring to Godsil and Royle (2001) for details). Moreover, A^{fl} and D^{fl} denote the adjacency and degree matrix between followers and leaders, respectively, i.e., $(A^{fl})_{ir} = a_{i(n+r)}$, $D^{fl} = \mathrm{diag}\{d_1^{fl}, \ldots, d_n^{fl}\}$, where $d_i^{fl} = \sum_{1\le r\le l} a_{i(n+r)} (1 \le i \le n)$.

To deal with random switching of multi-agent systems, we have to consider stochastic processes (referring to Chow and Teicher (1997), Norris (1997) and Ross (1983)). Given a probability space $(\Xi, \mathcal{F}, \mathbf{P})$. The elements of Ξ are called sample events. For $Q \in \mathcal{F}$, the indicator function $\chi_Q : \Xi \to R$ is defined by $\chi_Q(w) = 1$ if $w \in Q$, and $\chi_Q(w) = 0$ otherwise. $\mathbf{E}x$ denotes the expectation of random variable x. Let $\{\varphi_k, k = 0, 1, \ldots\}$ be an ergodic stationary sequence, and g an infinite dimensional Borel measurable function. Then $\{\xi_k, k = 0, 1, \ldots\}$ is also an ergodic stationary sequence if $\mathbf{E}|\xi_0| < +\infty$, where $\xi_k = g(\varphi_k, \varphi_{k+1}, \ldots)$. According to the strong law of large numbers of ergodic stationary sequence,

$$\lim_{k\to\infty} (\xi_0 + \xi_1 + \cdots + \xi_k)/(k + 1) = \mathbf{E}\xi_0 \quad a.s.$$

Let $\{\sigma(t), t \ge 0\}$ be a homogeneous irreducible continuous-time Markov chain taking values in a finite set $\mathcal{S} = \{1, \ldots, s^*\}$ of positive recurrent states. Define random variable sequence $t_0 = 0$,

$$t_{k+1} = \min\{t|t > t_k, \sigma(t) \ne \sigma(t_k)\}, \quad k = 0, 1, \ldots. \tag{1}$$

Then $\{t_{k+1} - t_k, k = 0, 1, \ldots\}$ are independent, conditional on $\{\sigma(t_k), k = 0, 1, \ldots\}$ and, for each $r \in \mathcal{S}$, there is a scalar $0 < \rho_r < \infty$[2] such that $t_{k+1} - t_k$ has the exponential distribution with parameter ρ_r, conditional on $\sigma(t_k) = r$. In addition, $t_k \to \infty$ as $k \to \infty$ with probability one. The embedded Markov chain is defined as $\{\sigma(t_k), k = 0, 1, \ldots\}$, which is homogeneous, irreducible and takes values in $\mathcal{S}$. Let $\varpi = (\varpi_{ij}) \in R^{s^*\times s^*}$ be its transition probability matrix. According to (1), $\varpi_{ii} = 0$ for all $i \in \mathcal{S}$. Moreover, let $\pi = (\pi_1, \ldots, \pi_{s^*})$ be its unique stationary distribution, i.e., $\mathbf{P}(\sigma(t_k) = r) = \mathbf{P}(\sigma(0) = r) = \pi_r$ for any k, where $\pi_r > 0$ for all $1 \le r \le s^*$. Here $\mathbf{P} = \mathbf{P}_\pi$ is the probability measure generated by the unique stationary distribution and transition probability matrix ϖ, and then under $\mathbf{P}$ the embedded Markov chain $\{\sigma(t_k), k = 0, 1, \ldots\}$ is an ergodic stationary sequence. $\mathbf{E} = \mathbf{E}_\pi$ is the expectation corresponding to $\mathbf{P}$. In fact, the obtained conclusions also hold for $\mathbf{P} = \mathbf{P}_{\bar\pi}$, where $\bar\pi$ is any given initial distribution.

Discrete-time Markovian random graphs were discussed in Matei et al. (2009), and here we give a corresponding concept for continuous-time cases.

Definition 1. Let $\mathcal{P} = \{G_r, r = 1, \ldots, s^*\}$ be a set of digraphs with n followers and l leaders. By a continuous-time Markovian random graph process we understand a map $\mathbf{G} : \mathcal{S} \to \mathcal{P}$ such that $\mathbf{G}(\sigma(t)) = G_{\sigma(t)}$ for any $t \ge 0$, where $\{\sigma(t), t \ge 0\}$ is a continuous-time homogeneous irreducible Markov chain taking values in a finite set $\mathcal{S} = \{1, \ldots, s^*\}$ of positive recurrent states.

In our multi-agent problem, the dynamic of leader i $(i = n + 1, \ldots, n + l)$ is expressed as:

$$\dot h_i = f_i(h, t), \quad h = (h_{n+1}^T, \ldots, h_{n+l}^T)^T \tag{2}$$

where $h_i \in R^m$ is the position of leader i, and $f_i(h, t) : R^{lm} \times R \to R^m$ is its velocity, piecewise continuous in (h, t). The dynamic of agent i $(i = 1, \ldots, n)$ is described by:

$$\dot x_i = v_i, \qquad \dot v_i = u_i \tag{3}$$

[2] The irreducibility of Markov chain implies that $\rho_r > 0$ for all $1 \le r \le s^*$. A state r for which $\rho_r = \infty$ means that it is instantaneously left once entered. Without loss of generality, in this paper we assume $\rho_r < \infty$ for all $1 \le r \le s^*$.

with

$$u_i(t) = -\gamma v_i(t) + \sum_{j\in\mathcal{N}_{il}(\mathbf{G}(\sigma(t)))} a_{ij}(t)(h_j(t) - x_i(t)) + \sum_{j\in\mathcal{N}_{if}(\mathbf{G}(\sigma(t)))} a_{ij}(t)[x_j(t) - x_i(t) + \alpha(v_j(t) - v_i(t))] \quad (4)$$

where $x_i, v_i, u_i \in R^m$ is the agent's position, velocity and control input, respectively. $\gamma > 0$ and $\alpha > 0$ are control parameters. Let $x = (x_1^T, \ldots, x_n^T)^T$ and $v = (v_1^T, \ldots, v_n^T)^T$. $\mathbf{G}$ is the continuous-time Markovian random graph process. $\{t_k, k = 0, 1, \ldots\}$ defined in (1) forms the switching time sequence and the interconnection topology between two successive switching times keeps invariant, that is, $\sigma(t) = \sigma(t_k)$ for $t \in [t_k, t_{k+1})$. Here $\mathbf{P}(t_{k+1}-t_k \le T|\sigma(t_k) = r) = 1 - e^{-\rho_r T}$ for $T > 0$.

The weights of arcs associated with digraph $G_{\sigma(t)}$ are denoted by $a_{ij}(t)$, i.e., $a_{ij}(t) = (A(G_r))_{ij}$ if $\sigma(t) = r$. These weights take values from a finite set and hence,

$$a_- \le a_{ij}(t) \le a^+ \quad (5)$$

for two positive constants a^+ and a_- if $a_{ij}(t) > 0$.

Here the velocities of leaders may be unavailable and $h_{n+1}, \ldots, h_{n+l}$ are the only measurable variables by the agents once they are connected to the leader set $\mathcal{L}$.

A set $\bar{\Omega} \subset R^m$ is said to be convex if $(1-\lambda)\vartheta_1 + \lambda\vartheta_2 \in \bar{\Omega}$ for any $\vartheta_1, \vartheta_2 \in \bar{\Omega}$ and $0 < \lambda < 1$. A function $f : R^m \to R$ is said to be convex if $f((1-\lambda)\omega_1 + \lambda\omega_2) \le (1-\lambda)f(\omega_1) + \lambda f(\omega_2)$ for any $\omega_1, \omega_2 \in R^m$ and $0 < \lambda < 1$. For a closed convex set K of a Hilbert space H, we can associate to any $\vartheta \in H$ a unique element $\pi_K(\vartheta) \in K$ satisfying $\|\vartheta - \pi_K(\vartheta)\|_2 = \inf_{\bar{\vartheta}\in K} \|\vartheta - \bar{\vartheta}\|_2$ (Rockafellar, 1972). Denote $\Omega = co\{h_{n+1}, \ldots, h_{n+l}\}$ as the polytope consisting of all finite convex combinations of the positions $h_{n+1}, \ldots, h_{n+l}$. $\Omega(t) = co\{h_{n+1}(t), \ldots, h_{n+l}(t)\}$ is a time-varying convex set if the leaders are moving. Define

$$d(\eta_i, \Omega) = \|\eta_i - \pi_\Omega(\eta_i)\|_2, \qquad d(\eta, \Omega^{\bar{n}}) = \max_{1\le i\le \bar{n}} d(\eta_i, \Omega),$$

where $\eta_i \in R^m$, $\eta = (\eta_1^T, \ldots, \eta_{\bar{n}}^T)^T$, $\bar{n} > 0$ is an integer.

Definition 2. The containment with respect to $\Omega(t)$ with bounded error can be solved in the expectation sense if for any initial condition, there are $\bar{M}_1 \ge 0, \bar{M}_2 \ge 0$ such that

$$\limsup_{t\to\infty} \mathbf{E}d(x(t), \Omega^n(t)) \le \bar{M}_1, \qquad \limsup_{t\to\infty} \mathbf{E}\|v(t)\|_2 \le \bar{M}_2.$$

The containment with respect to $\Omega(t)$ with bounded error can be solved almost surely if for any initial condition, there are $\hat{M}_1 \ge 0$, $\hat{M}_2 \ge 0$ such that almost surely

$$\limsup_{t\to\infty} d(x(t), \Omega^n(t)) \le \hat{M}_1, \qquad \limsup_{t\to\infty} \|v(t)\|_2 \le \hat{M}_2.$$

Furthermore, the containment with respect to $\Omega(t)$ is solved in the expectation sense and is solved almost surely if $\bar{M}_1 = 0$ and $\hat{M}_1 = 0$, respectively.

3. Basic analysis

In this section, we give basic analysis for the main results. Firstly, we give two basic assumptions.

Assumption 1. The set $\Omega(t)$ is bounded with a certain constant d^*, that is,

$$d^* = \sup_{t\ge 0} \sup_{\varrho_1,\varrho_2\in\Omega(t)} \|\varrho_1 - \varrho_2\|_2 < \infty. \quad (6)$$

Assumption 2. The leader set $\mathcal{L}$ is globally reachable in union digraph $\bigcup_{1\le r\le s^*} G_r$, where s^* is the number of the states of the considered Markov chain.

These assumptions are reasonable for multi-agent containment with switching topologies: without Assumption 1, $\Omega(t)$ may blow up quickly and cover all the agents without any requirements on the agent control; without Assumption 2, some agents may be separated from all leaders and form several sub-systems without any interconnections between them.

Rewrite system (3) in a compact form:

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -((L_\sigma^f + D_\sigma^{fl}) \otimes I_m)x \\ \quad -((\gamma I_n + \alpha L_\sigma^f) \otimes I_m)v + (A_\sigma^{fl} \otimes I_m)h. \end{cases} \quad (7)$$

Usually, the state matrix of the second-order system (7) is not Laplacian, unlike those of first-order systems. To overcome the difficulty, with $z = (y^T, h^T)^T$ and $y = (x^T, (x + \alpha v)^T)^T$, system (7) is equivalently transformed to

$$\dot{z}(t) = -(L_{\sigma(t)}^* \otimes I_m)z(t) + c(t), \quad (8)$$

where $c(t) = (0, \ldots, 0, f_{n+1}^T(h, t), \ldots, f_{n+l}^T(h, t))^T$ and

$$L^* = \begin{pmatrix} \frac{1}{\alpha} I_n & -\frac{1}{\alpha} I_n & 0 \\ \left(\frac{1}{\alpha} - \gamma\right) I_n + \alpha D^{fl} & \left(\gamma - \frac{1}{\alpha}\right) I_n + \alpha L^f & -\alpha A^{fl} \\ 0 & 0 & 0 \end{pmatrix}. \quad (9)$$

Lemma 1. *With a^+ defined in (5), L^* in (9) is Laplacian if*

$$\gamma \ge \frac{1}{\alpha} + \alpha l a^+. \quad (10)$$

This lemma can be easily verified since all off-diagonal elements of L^* are non-positive and all the diagonal elements are nonnegative with $L^*\mathbf{1} = 0$ with $\mathbf{1} = (1, \ldots, 1)^T$. In fact, similar results were obtained in the containment of second-order systems (referring to Cao and Ren (2010)) with suitably selected $v_i(t)$.

A nonnegative matrix is called stochastic if its each row sum equals to one, and for any $s > 0$, $\exp(-sL^*)$ is stochastic and has positive diagonal elements (Porfiri & Stilwell, 2007). Moreover, we have the following result, whose proof is quite easy and omitted here.

Lemma 2. *If $(p, q) \in \mathcal{E}$, then $(\exp(-sL^*))_{(n+p)(n+q)} > 0$ and $(\exp(-sL^*))_{p(n+q)} > 0$, where L^* is defined in (9).*

Clearly,

$$z(t_{k+1}) = (\exp(-(t_{k+1} - t_k)L_{\sigma(t_k)}^*) \otimes I_m)z(t_k) + b_{\sigma(t_k)}$$

with $b_{\sigma(t_k)} = \int_{t_k}^{t_{k+1}} \exp(-(t_{k+1} - s)L_{\sigma(t_k)}^* \otimes I_m)c(s)ds$. Define the transition matrix

$$\Phi(t_{k_2}, t_{k_1}) = \exp(-(t_{k_2+1} - t_{k_2})L_{\sigma(t_{k_2})}^*) \cdots \exp(-(t_{k_1+2} - t_{k_1+1})L_{\sigma(t_{k_1+1})}^*) \exp(-(t_{k_1+1} - t_{k_1})L_{\sigma(t_{k_1})}^*) \quad (11)$$

for $k_2 \ge k_1$. Then

$$z(t_{k_2+1}) = (\Phi(t_{k_2}, t_{k_1}) \otimes I_m)z(t_{k_1}) + b_{\sigma(t_{k_2})} + \sum_{k_1\le j<k_2} (\Phi(t_{k_2}, t_{j+1}) \otimes I_m)b_{\sigma(t_j)}. \quad (12)$$

The next lemma is given for the estimation analysis.

Lemma 3. *$d(\cdot, \bar{\Omega})$ is convex on R^m, where $\bar{\Omega} \subseteq R^m$ is closed and convex. Moreover, $d((B \otimes I_m)y(t) + (C \otimes I_m)h(t), \Omega^{2n}(t)) \le \|B\|_\infty d(y(t), \Omega^{2n}(t))$, where $B \in R^{2n\times 2n}$ and $C \in R^{2n\times l}$ are nonnegative matrices and each row sum of (B, C) is one.*

Proof. Since $\bar{\Omega}$ is convex, $(1-\lambda)\pi_{\bar{\Omega}}(\varsigma_1)+\lambda\pi_{\bar{\Omega}}(\varsigma_2)\in\bar{\Omega}$ for any $\varsigma_1,\varsigma_2\in R^m$ and $0<\lambda<1$. Thus,

$$\begin{aligned}
&d((1-\lambda)\varsigma_1+\lambda\varsigma_2,\bar{\Omega})\\
&\quad\le\|(1-\lambda)\varsigma_1+\lambda\varsigma_2-((1-\lambda)\pi_{\bar{\Omega}}(\varsigma_1)+\lambda\pi_{\bar{\Omega}}(\varsigma_2))\|_2\\
&\quad\le(1-\lambda)\|\varsigma_1-\pi_{\bar{\Omega}}(\varsigma_1)\|_2+\lambda\|\varsigma_2-\pi_{\bar{\Omega}}(\varsigma_2)\|_2\\
&\quad=(1-\lambda)d(\varsigma_1,\bar{\Omega})+\lambda d(\varsigma_2,\bar{\Omega}).
\end{aligned}$$

Furthermore, for any $1\le i\le 2n$,

$$\begin{aligned}
&d\left(\sum_{1\le j\le 2n}(B)_{ij}y_j(t)+\sum_{1\le j\le l}(C)_{ij}h_{n+j}(t),\Omega(t)\right)\\
&\quad\le\sum_{1\le j\le 2n}(B)_{ij}d(y_j(t),\Omega(t))+\sum_{1\le j\le l}(C)_{ij}d(h_{n+j}(t),\Omega(t))\\
&\quad=\sum_{1\le j\le 2n}(B)_{ij}d(y_j(t),\Omega(t))\le\|B\|_\infty d(y(t),\Omega^{2n}(t)).
\end{aligned}$$

Thus, the conclusion follows. □

4. Static leaders

In this section, we consider the target containment problem of a multi-agent system in the almost sure sense with static leaders. In this case, system (8) can be written as

$$\dot{z}=-(L_\sigma^*\otimes I_m)z. \tag{13}$$

Clearly, $z(t_{k+1})=(\exp(-(t_{k+1}-t_k)L^*_{\sigma(t_k)})\otimes I_m)z(t_k)$. By Lemma 3, we can easily obtain the next lemma.

Lemma 4. *With (10), $d(y(t),\Omega^{2n})$ is non-increasing along all the trajectories of system (13).*

Here is the main result for static leaders ($f_i=0,i\in\mathcal{L}$).

Theorem 1. *For system (2) and (3) with (10), the containment with respect to Ω can be solved almost surely if and only if Assumption 2 holds.*

Proof. Without loss of generality, suppose $m=1$ in the proof for notational simplicity. The necessity part is obvious, next we focus on the sufficiency part.

Since $\mathcal{L}$ is globally reachable in $\bigcup_{1\le r\le s^*}G_r$, for each $i\in\mathcal{I}$, there exists a path $i=i_0,i_1,\ldots,i_{p_i}$ from i to $i_{p_i}\in\mathcal{L}$ in $\bigcup_{1\le r\le s^*}G_r$, where $i_k\in\mathcal{I}$ $(0\le k\le p_i-1)$. Correspondingly, there exist $r_{i_k}\in\mathcal{S}$ $(0\le k\le p_i-1)$ such that $(i_k,i_{k+1})\in\mathcal{E}(G_{r_{i_k}})$. Set

$$\begin{aligned}
W_i&=W_i(s_{i_0},s_{i_1},\ldots,s_{i_{p_i-1}})=\exp(-s_{i_0}L^*_{r_{i_0}})\\
&\quad\cdot\exp(-s_{i_1}L^*_{r_{i_1}})\cdots\exp(-s_{i_{p_i-1}}L^*_{r_{i_{p_i-1}}}),
\end{aligned}\tag{14}$$

where $s_{i_k}>0$ and $L^*_{r_{i_k}}$ is the Laplacian matrix associated with $G_{r_{i_k}}$. By Lemma 2, $(\exp(-s_{i_0}L^*_{r_{i_0}}))_{i(n+i_1)}>0$ and $(\exp(-s_{i_k}L^*_{r_{i_k}}))_{(n+i_k)(n+i_{k+1})}>0$. Then $(W_i)_{i(n+i_{p_i})}>0$ and $(W_i)_{(n+i)(n+i_{p_i})}>0$. Denote W as

$$W(s_{1_0},\ldots,s_{1_{p_1-1}},\ldots,s_{n_0},\ldots,s_{n_{p_n-1}})=W_n\cdots W_1 \tag{15}$$

and let $N=\sum_{1\le i\le n}i_{p(i)}$ be the number of matrix exponentials on the right hand of (15). Obviously, for each $i=1,\ldots,n$, $(W)_{i(n+i_{p_i})}>0$ and $(W)_{(n+i)(n+i_{p_i})}>0$. Moreover, $\|(W)_{**}\|_\infty<1$ because W is stochastic. Since $\|(W)_{**}\|_\infty$ is continuous on bounded closed set $[T^*,T^{**}]^N$ with two positive numbers T^*,T^{**}, the upper and lower bound can be achieved. Therefore, there is $0<\beta<1$ such that for any $s_{i_k}\in[T^*,T^{**}]$ $(1\le i\le n,0\le k\le p_i-1)$, $\|(W)_{**}\|_\infty\le\beta$.

Denote $\mathbf{W}=\{W(s_{1_0},\ldots,s_{1_{p_1-1}},\ldots,s_{n_0},\ldots,s_{n_{p_n-1}})|s_{i_k}\in[T^*,T^{**}]\}$. Without loss of generality, we assume $\mathbf{P}(\|(\Phi(t_{N-1},0))_{**}\|_\infty\le\beta)>0$ with $\Phi(\cdot,\cdot)$ defined in (11). In fact, if $\varpi_{r_{i_{k+1}}r_{i_k}}>0$ and $\varpi_{r_ir_{(i+1)_{p_{i+1}-1}}}>0$ for all $1\le i\le n$ and $0\le k\le p_i-2$, then we have

$$\begin{aligned}
&\mathbf{P}(\|(\Phi(t_{N-1},0))_{**}\|_\infty\le\beta)\ge\mathbf{P}(\Phi(t_{N-1},0)\in\mathbf{W})\\
&\quad\ge\mathbf{P}(\mathcal{A},\mathcal{B})=\mathbf{P}(\mathcal{A}|\mathcal{B})\mathbf{P}(\mathcal{B})\\
&\quad=\pi_{r_{1_{p_1-1}}}\prod_{1\le i\le n}\left[\left(\prod_{0\le k\le p_i-2}\varpi_{r_{i_{k+1}}r_{i_k}}\left(e^{-\rho_{r_{i_{k+1}}}T^*}\right.\right.\right.\\
&\qquad\left.\left.\left.-\ e^{-\rho_{r_{i_{k+1}}}T^{**}}\right)\right)\varpi_{r_ir_{(i+1)_{p_{i+1}-1}}}\left(e^{-\rho_{r_i}T^*}-e^{-\rho_{r_i}T^{**}}\right)\right]\\
&\quad\triangleq p^*>0,
\end{aligned}$$

where $\varpi_{r_nr_{(n+1)_{p_{n+1}-1}}}=1$ and

$$\begin{aligned}
\mathcal{A}&=\{T^*\le t_1-t_0\le T^{**},\ldots,T^*\le t_N-t_{N-1}\le T^{**}\},\\
\mathcal{B}&=\{\sigma(t_0)=r_{1_{p_1-1}},\ldots,\sigma(t_{p_1-1})=r_1,\ldots,\\
&\qquad\sigma(t_{\sum_{1\le i\le n-1}p_i})=r_{n_{p_n-1}},\ldots,\sigma(t_{N-1})=r_n\}.
\end{aligned}$$

Otherwise, for some $i\in\mathcal{I}$ and $0\le q\le p_i-2$, $\varpi_{r_{i_{q+1}}r_{i_q}}=0$. Since the Markov chain is irreducible, there exist $i_{q_1},i_{q_2},\ldots,i_{q_{v_q}}\in\mathcal{S}$ such that $\varpi_{r_{i_{q+1}}r_{i_{q_1}}}>0,\varpi_{r_{i_{q_1}}r_{i_{q_2}}}>0,\ldots,\varpi_{r_{i_{q_{v_q}}}r_{i_q}}>0$. Set $Z_{r_{i_{q+1}}r_{i_q}}=\exp(-s_{i_{v_q}}L^*_{r_{i_{v_q}}})\cdots\exp(-s_{i_{q_2}}L^*_{r_{i_{q_2}}})\exp(-s_{i_{q_1}}L^*_{r_{i_{q_1}}})$. On the right hand of (14), we multiply matrix $Z_{r_{i_{q+1}}r_{i_q}}$ between $\exp(-s_{i_{q+1}}L^*_{r_{i_{q+1}}})$ and $\exp(-s_{i_q}L^*_{r_{i_q}})$ and the new product matrix can still be denoted as W_i for simplicity. Moreover, if there is $1\le j\le n$ such that $\varpi_{r_jr_{(j+1)_{p_{j+1}-1}}}=0$, then similarly we obtain a new product matrix still denoted as W for simplicity.

Take $\xi_k=\chi_{\{\|(\Phi(t_{k+N-1},t_k))_{**}\|_\infty\le\beta\}}$. $\{\xi_k,k=0,1,\ldots\}$ and then $\{\xi_k,k=0,N,2N,\ldots\}$ are ergodic stationary sequences. Based on the strong law of large numbers,

$$\begin{aligned}
&\lim_{k\to\infty}\frac{\xi_0+\xi_N+\cdots+\xi_{kN}}{k+1}=\mathbf{E}\xi_0\\
&\quad=\mathbf{P}(\|(\Phi(t_{N-1},0))_{**}\|_\infty\le\beta)\ge p^*>0\quad a.s.
\end{aligned}$$

Therefore, for almost all sample event w, there is an increasing sequence $\{k_\upsilon=k_\upsilon(w),\upsilon=1,2,\ldots\}$ such that $\xi_{k_\upsilon N}=1$, namely, $\|(\Phi(t_{(k_\upsilon+1)N-1},t_{k_\upsilon N}))_{**}\|_\infty\le\beta$. By Lemmas 3 and 4,

$$\begin{aligned}
&d(y(t_{(k_\upsilon+1)N}),\Omega^{2n})\le\beta d(y(t_{k_\upsilon N}),\Omega^{2n})\\
&\quad\le\beta d(y(t_{(k_{\upsilon-1}+1)N}),\Omega^{2n})\le\beta^2d(y(t_{k_{\upsilon-1}N}),\Omega^{2n})\\
&\quad\le\cdots\le\beta^\upsilon d(y(t_{k_1N}),\Omega^{2n})\le\beta^\upsilon d(y_0,\Omega^{2n}).
\end{aligned}$$

As a result, for almost all the trajectories, there is a subsequence $\{k_1,k_2,\ldots\}$ such that $d(y(t_{(k_\upsilon+1)N}),\Omega^{2n})\to0$ as $\upsilon\to\infty$. Thus, $\lim_{t\to\infty}d(y(t),\Omega^{2n})=0$ *a.s.* by Lemma 4. In addition, $d(x(t),\Omega^n)\le d(y(t),\Omega^{2n})$ implies $\lim_{t\to\infty}d(x(t),\Omega^n)=0$ *a.s.* We complete the sufficiency part by $\limsup_{t\to\infty}\|v(t)\|_2\le\hat{M}_2=\frac{1}{\alpha}\sqrt{n}d^*$ *a.s.* since

$$|v_i|\le\frac{1}{\alpha}\max_{1\le i\ne j\le 2n}|y_i-y_j|\le\frac{1}{\alpha}(2d(y,\Omega^{2n})+d^*) \tag{16}$$

with d^* defined in (6). □

5. Moving leaders

In this section, we consider the containment of second-order agent dynamics with moving leaders.

Assume random variables $\{\sigma(t_k), k = 0, 1, \ldots\}$ are independent and identically distributed (i.i.d.) with $\mathbf{P}(\sigma(t_0) = r) = \pi_r$ $(1 \le r \le s^*)$, which is a special class of Markov chain. In this case, $\{t_{k+1} - t_k, k = 0, 1, \ldots\}$ are i.i.d. and $\mathbf{E}t_1 = \sum_{1\le r\le s^*} \frac{\pi_r}{\rho_r}$. We first show a lemma on the distance estimation.

Lemma 5. *$d(\eta+\nu, \Omega^{\bar{n}}) \le d(\eta, \Omega^{\bar{n}}) + \|\nu\|_2$ with $\nu, \eta \in R^{\bar{n}m}$ and an integer $\bar{n} > 0$. Moreover, $d(\eta, \Omega^{\bar{n}}(t_{k+1})) \le d(\eta, \Omega^{\bar{n}}(t_k)) + (t_{k+1} - t_k)M$ if $\|f_j\|_2 \le M$ for all $j \in \mathcal{L}$.*

Proof. Set $\varepsilon_i = d(\eta_i, \Omega)$ and $B(\Omega, \varepsilon_i) = \{\varsigma | d(\varsigma, \Omega) \le \varepsilon_i\}$ for some $1 \le i \le \bar{n}$. Then $d(\eta_i + \nu_i, \Omega(t)) \le d(\eta_i + \nu_i, B(\Omega(t), \varepsilon_i)) + \varepsilon_i \le \|\nu_i\|_2 + d(\eta_i, \Omega(t))$ if $\eta_i + \nu_i \in R^m \setminus B(\Omega(t), \varepsilon_i)$, or $d(\eta_i+\nu_i, \Omega(t)) \le \varepsilon_i \le d(\eta_i, \Omega(t)) + \|\nu_i\|_2$ if $\eta_i + \nu_i \in B(\Omega(t), \varepsilon_i)$.

Obviously, $\pi_{\Omega(t_k)}(\eta_i)$ can be expressed as the convex combination of vertexes of $\Omega(t_k)$. Let $\pi_{\Omega(t_k)}(\eta_i) = \sum_{1\le r\le l} \theta_r h_{n+r}(t_k)$, where $0 \le \theta_r \le 1$, $\sum_{1\le r\le l} \theta_r = 1$. It is not hard to find that, for any k and $1 \le r \le l$, $\|h_{n+r}(t_k) - h_{n+r}(t_{k+1})\|_2 \le (t_{k+1} - t_k)M$. Therefore,

$$\begin{aligned}
d(\eta_i, \Omega(t_{k+1})) &\le \Big\|\eta_i - \sum_{1\le r\le l} \theta_r h_{n+r}(t_{k+1})\Big\|_2 \\
&\le \Big\|\eta_i - \sum_{1\le r\le l} \theta_r h_{n+r}(t_k)\Big\|_2 \\
&\quad + \Big\|\sum_{1\le r\le l} \theta_r h_{n+r}(t_k) - \sum_{1\le r\le l} \theta_r h_{n+r}(t_{k+1})\Big\|_2 \\
&\le d(\eta_i, \Omega(t_k)) + \sum_{1\le r\le l} \theta_r \|h_{n+r}(t_k) - h_{n+r}(t_{k+1})\|_2 \\
&\le d(\eta_i, \Omega(t_k)) + (t_{k+1} - t_k)M,
\end{aligned}$$

which implies the conclusion. □

Next, we introduce a result in the expectation sense.

Theorem 2. *For system (2) and (3) with (10) and Assumption 1, the containment with respect to $\Omega(t)$ with bounded error can be solved in the expectation sense if the velocities of the leaders are bounded and Assumption 2 holds.*

Proof. Still take W with the form of (15) given in Theorem 1. By Lemmas 3, 5 and (12), we have

$$\begin{aligned}
d(y(t_N), \Omega^{2n}(t_N)) &= d(z(t_N), \Omega^{2n+l}(t_N)) \\
&\le d(z(t_N), \Omega^{2n+l}(0)) + t_N M \\
&\le d(\Phi(t_{N-1}, 0)z(0), \Omega^{2n+l}(0)) + t_N M \\
&\quad + \Big\|\sum_{0\le j<N-1} \Phi(t_{N-1}, t_{j+1}) b_{\sigma(t_j)} + b_{\sigma(t_{N-1})}\Big\|_2 \\
&\le \|(\Phi(t_{N-1}, 0))_{**}\|_\infty d(y(0), \Omega^{2n}(0)) + t_N M_0,
\end{aligned}$$

where $M_0 = (1 + \sqrt{2n+l})M$ and M is the upper bound of the velocities defined in Lemma 5. It is not hard to find that $\mathbf{E}\|(\Phi(t_{N-1}, 0))_{**}\|_\infty \le 1 - (1-\beta)p^*$ and $\mathbf{E}t_N = N(\sum_{1\le r\le s^*} \frac{\pi_r}{\rho_r})$. Then $\mathbf{E}d(y(t_N), \Omega^{2n}(t_N)) \le (1 - (1-\beta)p^*)\mathbf{E}d(y(0), \Omega^{2n}(0)) + N(\sum_{1\le r\le s^*} \frac{\pi_r}{\rho_r})M_0$.

Similarly, for $k = 2, 3, \ldots,$

$$\begin{aligned}
\mathbf{E}d(y(t_{kN}), \Omega^{2n}(t_{kN})) &\le (1 - (1-\beta)p^*)\mathbf{E}d(y(t_{(k-1)N}), \\
&\quad \Omega^{2n}(t_{(k-1)N})) + N\left(\sum_{1\le r\le s^*} \frac{\pi_r}{\rho_r}\right) M_0.
\end{aligned}$$

Due to $0 < 1 - (1-\beta)p^* < 1$,

$$\limsup_{k\to\infty} \mathbf{E}d(y(t_{kN}), \Omega^{2n}(t_{kN})) \le \frac{N\left(\sum_{1\le r\le s^*} \frac{\pi_r}{\rho_r}\right) M_0}{(1-\beta)p^*}.$$

In addition, for any $t \in [t_{kN}, t_{(k+1)N})$, $d(y(t), \Omega^{2n}(t)) \le d(y(t_{kN}), \Omega^{2n}(t_{kN})) + (t_{(k+1)N} - t_{kN})M_0$, and then

$$\limsup_{t\to\infty} \mathbf{E}d(y(t), \Omega^{2n}(t)) \le \bar{M}_1 = \frac{(1 + (1-\beta)p^*)N\left(\sum_{1\le r\le s^*} \frac{\pi_r}{\rho_r}\right)M_0}{(1-\beta)p^*}. \quad (17)$$

Moreover, by (16) and $d(x(t), \Omega^n(t)) \le d(y(t), \Omega^{2n}(t))$, the conclusion follows with $\bar{M}_1$ defined in (17) and $\bar{M}_2 = \frac{1}{\alpha}\sqrt{n}$ $(2\bar{M}_1 + d^*)$. □

Remark 1. From the proof of Theorem 2, Assumption 1 is not required to prove $\limsup_{t\to\infty} \mathbf{E}d(x(t), \Omega^n(t)) \le \bar{M}_1$. Moreover, by the conditions of Theorem 2 and $\|f_i(h, t)\|_2 \le M_t$ for all $i \in \mathcal{L}$ with $\lim_{t\to\infty} M_t = 0$, we can solve the containment with respect to $\Omega(t)$ in the expectation sense.

Remark 2. Theorem 2 is consistent with Theorem 1. For static leaders ($f_i = 0, i \in \mathcal{L}$), $\lim_{t\to\infty} \mathbf{E}d(y(t), \Omega^{2n}) = 0$ from (17) by taking $M = 0$, and then $\mathbf{E}d(y(k), \Omega^{2n}) \to 0$ as $k \to \infty$. Clearly, $\{d(y(k), \Omega^{2n}), k = 0, 1, \ldots\}$ converges in probability to 0. Then there is a subsequence $\{d(y(k_\upsilon), \Omega^{2n}), \upsilon = 0, 1, \ldots\}$ which converges to 0 almost surely (Chow & Teicher, 1997). By Lemma 4 and the relation $d(x(t), \Omega^n) \le d(y(t), \Omega^{2n})$, we have that $\lim_{t\to\infty} d(x(t), \Omega^n) = 0$ almost surely.

Assumption 2 is not necessary in Theorem 2. If the leaders move in a bounded region, followers may keep bounded even if they may not be connected to the leaders. In this case, the containment with a certain bounded error is solved. However, the containment error bound under Assumption 2 may be much less than that without Assumption 2, as shown in the following example.

Example 1. Consider a system with one follower and two leaders in R. The dynamics of two leaders are in the forms of $\dot{h}_i = \cos t$, $i = 2, 3$ with initial conditions $h_2(0), h_3(0)$. Their solutions are $h_i(t) = \sin t + h_i(0)$. Obviously, Assumption 1 holds. Choose $\alpha = 1$, $\gamma \ge 3$ and all nonzero weights are one. If $\mathcal{E}(G_i)$ is empty for $i = 2, 3$, then by (4) the dynamic of the follower is $\dot{x}_1 = v_1$, $\dot{v}_1 = -\gamma v_1$ with initial condition $(x_1(0), v_1(0))$. Clearly, $x_1(t) = x_1(0) + (1 - e^{-\gamma t})v_1(0)/\gamma$. Hence, the containment error bound is $\max\{|x_1(0) + \frac{1}{\gamma}v_1(0) - h_i(0) + 1|, |x_1(0) + \frac{1}{\gamma}v_1(0) - h_i(0) - 1|, i = 2, 3\}$ without Assumption 2. If $\mathcal{E}(G_i) = \{(1, 2)\}$ for $i = 2, 3$, then Assumption 2 holds and the follower dynamic becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{v}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix} \begin{pmatrix} x_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin t + h_2(0) \end{pmatrix}.$$

We can easily obtain the error

$$\begin{aligned}
\limsup_{t\to\infty} d(x_1(t), \Omega(t)) &\le \max\{|\bar{\lambda}|(1 + |h_2(0)|) \\
&\quad + |h_i(0) + 1|, |\bar{\lambda}|(1 + |h_2(0)|) + |h_i(0) - 1|, i = 2, 3\},
\end{aligned}$$

where $\bar{\lambda} < 0$ is the largest eigenvalue of system matrix. If $|x_1(0) + \frac{1}{\gamma}v_1(0) - h_2(0)|$ and $|x_1(0) + \frac{1}{\gamma}v_1(0) - h_3(0)|$ are sufficiently large, the error bound with Assumption 2 is much less than that without Assumption 2.

Next, we give another assumption to show when the sufficient topology condition becomes necessary.

Assumption 3. $f_i = f_j$ $(i, j \in \mathcal{L})$ are bounded $\limsup_{t\to\infty} \|\int_0^t f_i(h, s)ds\|_2 = \infty$.

It is easy to see Assumption 3 implies Assumption 1.

Theorem 3. *For system (2) and (3) with (10) and Assumption 3, the containment with respect to $\Omega(t)$ with bounded error can be solved in the expectation sense if and only if Assumption 2 holds.*

Proof. The sufficiency has been proved in Theorem 2. Here we only prove the necessity. If $\mathcal{L}$ is not globally reachable in $\bigcup_{1\le r\le s^*} G_r$, we define the nonempty set $\mathcal{I}_1 = \{i \in \mathcal{I} \mid$ there is no path from i to some leader in $\mathcal{L}$ in $\bigcup_{1\le r\le s^*} G_r\}$. Thus, all agents of $\mathcal{I}_1$ form a subsystem without connection to $\mathcal{V} \setminus \mathcal{I}_1$. Let $n_1 = |\mathcal{I}_1|$ and $\bar{y} = (x_{i_1}^T, \ldots, x_{i_{n_1}}^T; x_{i_1}^T + \alpha v_{i_1}^T, \ldots, x_{i_{n_1}}^T + \alpha v_{i_{n_1}}^T)^T$, where $i_j \in \mathcal{I}_1$ $(1 \le j \le n_1)$. Based on (4), we can find a Laplacian $\bar{L}$ such that $\dot{\bar{y}}(t) = -(\bar{L}_{\sigma(t)} \otimes I_m)\bar{y}(t)$. Therefore, $\bar{y}(t) = (\exp(-(t-t_k)\bar{L}_{\sigma(t)}) \otimes I_m)\bar{y}(t_k)$. Obviously, $\|\bar{y}(t) - \bar{y}(0)\|_2 \le \sqrt{2n_1}\max_{1\le i\ne j\le 2n_1}\|\bar{y}_i(0) - \bar{y}_j(0)\|_2$. Let $\pi_{\Omega(t)}(\bar{y}_i(t)) = \sum_{1\le r\le l}\theta_r h_{n+r}(t)$, where $1 \le i \le 2n_1$, $0 \le \theta_r \le 1$, and $\sum_{1\le r\le l}\theta_r = 1$. Then

$$\begin{aligned} k(t) &= \left\| \sum_{1\le r\le l}\theta_r h_{n+r}(t) - \sum_{1\le r\le l}\theta_r h_{n+r}(0)\right\|_2 \\ &\le \left\| \sum_{1\le r\le l}\theta_r h_{n+r}(t) - \bar{y}_i(t)\right\|_2 + \|\bar{y}_i(t) - \pi_{\Omega(0)}(\bar{y}_i(t))\|_2 \\ &\quad + \left\|\pi_{\Omega(0)}(\bar{y}_i(t)) - \sum_{1\le r\le l}\theta_r h_{n+r}(0)\right\|_2 \\ &\le d(\bar{y}_i(t), \Omega(t)) + d(\bar{y}_i(t), \Omega(0)) + d^* \end{aligned}$$

where $k(t) = \| \int_0^t f_i(h,s)ds\|_2$. From Lemma 5

$$\begin{aligned} d(\bar{y}(t), \Omega^{2n_1}(t)) &\ge k(t) - d(\bar{y}(t), \Omega^{2n_1}(0)) - d^* \\ &\ge k(t) - d(\bar{y}(0), \Omega^{2n_1}(0)) - d^* \\ &\quad - \sqrt{2n_1}\max_{1\le i\ne j\le 2n_1}\|\bar{y}_i(0) - \bar{y}_j(0)\|_2. \end{aligned}$$

On one hand, $\limsup_{t\to\infty} \mathbf{E}d(\bar{y}(t), \Omega^{2n_1}(t)) = \infty$, and then $\limsup_{t\to\infty} \mathbf{E}d(y(t), \Omega^{2n}(t)) = \infty$. On the other hand, again by Lemma 5,

$$\begin{aligned} d(y(t), \Omega^{2n}(t)) &= \max\{d(x(t), \Omega^n(t)), d(x(t) + \alpha v(t), \Omega^n(t))\} \\ &\le d(x(t), \Omega^n(t)) + \alpha\|v(t)\|_2. \end{aligned}$$

Consequently, $\limsup_{t\to\infty} \mathbf{E}d(y(t), \Omega^{2n}(t)) \le \bar{M}_1 + \alpha\bar{M}_2$. Thus, the contradiction leads to the conclusion. □

Here is an example for illustration.

Example 2. Consider a system consisting of three followers 1, 2, 3 and two leaders 4, 5 in R. For $\{\sigma(t_k), k = 0, 1, \ldots\}$, suppose $\{t_{k+1} - t_k, k = 0, 1, \ldots\}$ are independent and have the identical exponential distribution with $\rho_k = 1$. Here $s^* = 3$ and the transition probability matrix corresponding to the embedded Markov chain is

$$\varpi = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

The arc sets of graphs G_1, G_2, G_3 are $\{(2,3), (1,4)\}$, $\{(1,3), (2,4)\}$, $\{(3,2), (3,5)\}$, respectively. All nonzero weights are one. Choose $\alpha = 1$ and $\gamma = 3$ in the control. The three followers track the two moving leaders with the same velocity $f_4 = f_5 = 0.5$, as showed in Fig. 1, where we find the tracking error is bounded between the three followers and the moving segment with two leaders as its vertexes.

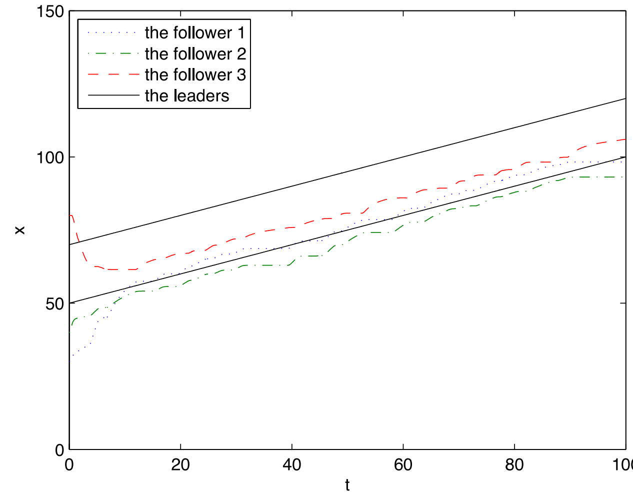

Fig. 1. Track moving leaders.

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