Distributed Surrounding Design of Target Region With Complex Adjacency Matrices

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Abstract—**In this technical note, we consider the distributed surrounding of a convex target set by a group of agents with switching communication graphs. We propose a distributed controller to surround a given set with the same distance and desired projection angles specified by a complex-value adjacency matrix. Under mild connectivity assumptions, we give results in both consistent and inconsistent cases for the set surrounding in a plane. Also, we provide sufficient conditions for the multi-agent coordination when the convex set contains only the origin.**

Index Terms—**Complex weights, joint connection, multi-agent systems, set surrounding.**

I. Introduction

The distributed coordination and control of multi-agent systems has been investigated from various perspectives due to its various applications. After the study of consensus or formation of multi-agent systems [2]–[7], much attention has been paid to set coordination problems of multi-agent systems. Among the studies of multi-agent set coordination, distributed containment control has achieved much, which makes agents reach a convex set maybe spanned by multiple leaders [9]–[12]. Moreover, some results were obtained to control a group of agents in order to protect or surround a convex target set. For example, the distributed controller was designed for the agents to surround all stationary leaders in the convex hull spanned by the agents in [17], while a model was provided for multiple robots to protect a target region [1]. However, many theoretical problems to surround a target set remain to be solved.

On the other hand, complex Laplacians or rotation matrices have been applied to consensus and formation (see [20]–[22]), partially because the complex representation may significantly simplify the analysis when the state space is a plane. Formation control for directed acyclic graphs with complex Laplacians and related stability analysis were discussed in [20], while new methods were developed for pattern formation with complex-value elements in [21].

The objective of this technical note is to study the distributed set surrounding design based on complex adjacency matrices, that is, to design a distributed protocol to make a group of agents protect/surround a convex set in a plane. We first propose a distributed controller to make all agents achieve the set projection with the same distance and different projection angles specified by a given complex-value adjacency matrix. For uniformly jointly strongly connected undirected graph and fixed strongly connected graph, we provide the initial conditions guaranteeing that all agents will not converge to the set. Then we investigate the special case when the set becomes the origin, with a necessary and sufficient condition in the fixed strongly connected graph case. In addition, our results also extend some existing ones including the consensus [2], [3] and bipartite consensus [13].

The contributions of this technical note include: 1) we proposed a distributed controller to solve the set surrounding problem under the switching communication graphs; 2) we characterize the relationship between the consistency of directed cycles of the configuration graph and the system dynamic behavior, or roughly speaking, the consistent cycles produce the consistent case, while inconsistent cycles yield the inconsistent case; 3) we extend some existing results of consensus and bipartite consensus when the set contains only one point.

The technical note is organized as follows. Section II gives preliminary knowledge and the problem formulation. Section III provides the main results for the distributed set surrounding problems and then considers an important special case when the target becomes the origin. Then Section IV gives a numerical example for illustration. Finally, Section V shows some concluding remarks.

Notation: $\mathbb{R}$ and $\mathbb{C}$ denotes the real field and complex field, respectively; $|\cdot|$ denotes the modulus of a complex number or the number of elements in a set; $P_X(\cdot)$ denotes the projection operator onto the closed convex set X; z^p denotes the projection vector of point z onto X, i.e., $z^p = z - P_X(z)$; $|\cdot|_X$ denotes the distance between a point and X, i.e., $|z|_X = |z - P_X(z)|$; $\iota = \sqrt{-1}$ denotes the imaginary unit; $\angle z$ denotes the argument of complex number z; $\langle \cdot, \cdot \rangle$ denotes the inner product of two complex numbers, i.e., $\langle a_1 + a_2\iota, b_1 + b_2\iota \rangle = a_1 b_1 + a_2 b_2$.

II. Preliminaries and Formulation

In this section, we first introduce preliminary knowledge and then formulate the distributed set surrounding problem.

A. Preliminaries

A digraph (or directed graph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of node set $\mathcal{V} = \{1, 2, \ldots, n\}$ and arc set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ [18]. A weak path in digraph $\mathcal{G}$ is an alternating sequence $i_1 e_1 i_2 e_2 \cdots i_k e_k i_{k+1}$ of nodes $i_r, r = 1, \ldots, k+1$ and arcs $e_r = (i_r, i_{r+1}) \in \mathcal{E}$ or $e_r = (i_{r+1}, i_r) \in \mathcal{E}, r = 1, \ldots, k$; if $e_r = (i_r, i_{r+1})$ for all r, the weak path becomes a directed path; if $i_1 = i_{k+1}$, the weak path is called a weak cycle; A weak cycle containing a directed path is called a directed cycle. Digraph $\mathcal{G}$ is said to be weakly strongly connected if there exists a weak path in $\mathcal{G}$ between every pair of nodes in $\mathcal{V}$, and strongly connected if there exists a directed path in $\mathcal{G}$ between every pair of nodes in $\mathcal{V}$. Moreover, $\mathcal{G}$ is undirected if $(i,j) \in \mathcal{E}$ is equivalent to $(j,i) \in \mathcal{E}$. Undirected graph (digraph) $\mathcal{G}$ is said to be a (directed) tree if there is one node such that there is one and only one (directed) path from any other node to this node. Undirected graph (digraph) $\mathcal{G}$ is said to contain a (directed) spanning tree if it has a (directed) tree containing all nodes of $\mathcal{G}$ as its subgraph. Here we assume $\mathcal{G}$ contains no self-loop, i.e., $(i,i) \notin \mathcal{E}, i \in \mathcal{V}$.

Consider a multi-agent system consists of n agents. Let $\sigma : [0, \infty) \to \mathcal{Q}$ be a piecewise constant function to describe the switching

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graph process with $\mathcal{Q}$ the index set of all possible digraphs on $\mathcal{V}$. Denote a switching graph with signal σ as $\mathcal{G}_\sigma = (\mathcal{V}, \mathcal{E}_\sigma)$, which is called a *communication* graph to describe the (time-varying) communication between the agents (regarded as nodes) with taking its connection weight $a_{ij} = 1$ if $(i, j) \in \mathcal{E}_\sigma$ for simplicity. Denote $\mathcal{G}_\sigma([t_1, t_2))$ as the union graph with node set $\mathcal{V}$ and arc set $\bigcup_{t_1 \le t < t_2} \mathcal{E}_{\sigma(t)}, 0 \le t_1 < t_2$. The switching digraph $\mathcal{G}_\sigma$ is uniformly jointly strongly connected (*UJSC*) if there exists $T > 0$ such that $\mathcal{G}_\sigma([t, t+T))$ is strongly connected for any $t \ge 0$. As usual, we assume there is a dwell time as the lower bound between two consecutive switching moments.

The Dini derivative of a continuous function $f : (a, b) \to \mathbb{R}$ at $t \in (a, b)$ is defined as follows:

$$D^+ f(t) = \limsup_{s \to 0^+} \frac{f(t+s) - f(t)}{s}.$$

Clearly, f is non-increasing on (a, b) if $D^+ f(t) \le 0, \forall t \in (a, b)$. The following result can be found in [16].

Lemma 1: Let $f_i(t, x) : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}, i = 1, \ldots, M$ be continuously differentiable and $f(t, x) = \max_{1 \le i \le M} f_i(t, x)$. Then $D^+ f(t, x(t)) = \max_{i \in \mathcal{I}(t)} \dot{f}_i(t, x(t))$, where $\mathcal{I}(t) = \{i | f_i(t, x(t)) = f(t, x(t)), 1 \le i \le M\}$.

As we know, a set K is said to be convex if $(1-\lambda) z_1 + \lambda z_2 \in K$ whenever $z_1, z_2 \in K$ and $0 \le \lambda \le 1$. Moreover, let $P_K(\cdot) : \mathbb{C} \to K$ be the projection operator onto closed convex set K, i.e., $P_K(z)$ is the unique element in K satisfying $\inf_{y \in K} |z - y| = |z - P_K(z)| := |z|_K$ [14].

B. Problem Formulation

Consider the n agents described by the first-order integrator

$$\dot{x}_i(t) = u_i(t), \qquad i = 1, \ldots, n, \tag{1}$$

where $x_i, u_i \in \mathbb{C}$ are the state and control input of agent i in the plane, respectively. Consider a 2-dimensional bounded closed convex set $X \subseteq \mathbb{R}^2$ to be surrounded. For a desired surrounding configuration or pattern, we need to assign the desired relative projection angles between the agents when they surround X. To this end, we give a complex-value adjacency matrix $W = (w_{ij}) \in \mathbb{C}^{n \times n}$ to describe the desired relative angles of projections for agents to X as follows: $w_{ii} = 0, i = 1, \ldots, n$ and either $|w_{ij}| = 1$ or $w_{ij} = 0$ for $i \ne j$. In this way, we get a digraph $\mathcal{G}^w = (\mathcal{V}, \mathcal{E}^w)$ with $\mathcal{E}^w = \{(i, j) | w_{ij} \ne 0\}$, which is called a *configuration* graph. Meanwhile, w_{ij} is called the *configuration weight* of arc $(i, j) \in \mathcal{E}^w$. A weak cycle $i_1 e_1 i_2 e_2 \cdots i_k e_k i_1$ in $\mathcal{G}^w$ is said to be consistent if

$$\prod_{r=1}^{k} w(e_r) = 1,$$

where $w(e_r) = w_{i_r i_{r+1}}$ for $e_r = (i_r, i_{r+1})$, $w(e_r) = w_{i_{r+1} i_r}^{-1}$ for $e_r = (i_{r+1}, i_r)$; otherwise, it is said to be inconsistent. Clearly, $w_{ij} w_{ji} = 1$ in the consistent case when $w_{ij} \ne 0$ and $w_{ji} \ne 0$.

Remark 1: Although no convex set gets involved in the control design in multi-agent formation [6], [21], its design is directly based on the desired formation configuration determined by the desired relative distances or positions. Sometimes, the desired formation can be described by a set of desired relative position vectors d_{ij} to show the desired position of agent j relative to that of agent i for $i, j = 1, \ldots, n$. In this case, for a given weak cycle $i_1 e_1 i_2 e_2 \cdots i_k e_k i_1$, we also have the consistent case with $\sum_{r=1}^{k} d(e_r) = 0$, and inconsistent case with $\sum_{r=1}^{k} d(e_r) \ne 0$, where $i_{k+1} = i_1$, $d(e_r) = d_{i_r i_{r+1}}$ for $e_r = (i_r, i_{r+1})$, $d(e_r) = -d_{i_{r+1} i_r}$ for $e_r = (i_{r+1}, i_r)$. It is known that the formation may fail in the inconsistent case. In our problem, the desired relative projection angles are described by w_{ij} to achieve the desired surrounding configuration, which plays a similar role as d_{ij} in the formation. Therefore, in both formation and surrounding problems, the agents' indexes are given in the desired configuration.

In this technical note, we consider how to surround the given set X with the same distance by the n agents from different projection angles (that is, the rotation angles of projection vectors) specified by W. To be strict, we introduce the following definition.

Definition 1: The distributed set surrounding is achieved for system (1) with a distributed control u_i if, for any initial condition $x_i(0), 1 \le i \le n$

$$\lim_{t \to \infty} w_{ij} \left(x_j(t) - P_X\left(x_j(t)\right)\right) - \left(x_i(t) - P_X\left(x_i(t)\right)\right) = 0$$

for $(i, j) \in \mathcal{E}^w$.

In fact, there are two cases for the set surrounding:

- **Consistent case**: All agents surround the convex set X with the same nonzero distance to X and desired projection angles between each other determined by the entries w_{ij} of W.
- **Inconsistent case**: all agents converge to the convex set X.

In what follows, we will show: if the weights given in the configuration graph are inconsistent, the inconsistent case appears; if the weights are consistent, we can achieve the consistent case somehow.

Remark 2: Different from the surrounding formulation given in [17], the agents in our problem not only surround the target set but also keep the same distance from the target set (potentially for balance or coordination concerns). Additionally, the inconsistent case resulting from the inconsistency of configuration graph is related to containment problems [10]–[12].

In practice, node i may not receive the information from node j sometimes due to communication failure or energy saving. Denote the set of all arcs transiting information successfully at time t as $\mathcal{E}_{\sigma(t)}$, which is a subset of $\mathcal{E}^w$, and the resulting graph is $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$, which is the *communication* graph of the multi-agent system. Note that the configuration graph $\mathcal{G}^w$ shows the desired relative projection angles of the agents, while the communication graph $\mathcal{G}_\sigma$, a subgraph of $\mathcal{G}^w$, describes the communication topology of the agents. Let $\mathcal{N}_i = \{j | (i, j) \in \mathcal{E}^w\}$ and $\mathcal{N}_i(\sigma(t)) \subseteq \mathcal{N}_i$ denote the neighbor set of node i in communication graph $\mathcal{G}_{\sigma(t)}$. Then we take the following control:

$$u_i(t) = \sum_{j \in \mathcal{N}_i(\sigma(t))} \left[w_{ij} (x_j(t) - P_X(x_j(t))) - (x_i(t) - P_X(x_i(t))) \right]. \tag{2}$$

As usual, in the design of controller (2), agents only count in the received information from their neighbors.

Remark 3: Let us check the role of complex-value adjacency matrix W. In Definition 1, the complex-value configuration weight $w_{ij} (= e^{\alpha_{ij} \iota})$ indicates that α_{ij} is the desired angle difference between projection vector of agent i onto set X and that of agent j. Because $|w_{ij}| = 1$ for $(i, j) \in \mathcal{E}^w$, $w_{ij} x_j^p(t) - x_i^p(t) \to 0$ implies $|x_i(t)|_X - |x_j(t)|_X \to 0$, and therefore, all agents will have the same distance to the convex set when the (consistent) set surrounding is achieved. If $|w_{ij}| \ne 1$, we may get the set surrounding with different distances from the agents to the target set.

III. Main Results

In this section, we will solve the following basic surrounding problems: (i) How to design distributed controllers to achieve the set surrounding? (ii) What initial conditions can guarantee the consistent case? (iii) What happens when the target set consists of only one point?

Before we study the set surrounding problem, we first show that the consistent case of the set surrounding problem is well-defined, which can be achieved in some situations.

Theorem 1: Consider a complex-value adjacency matrix W and the resulting configuration graph $\mathcal{G}^w = (\mathcal{V}, \mathcal{E}^w)$. If all weak cycles of $\mathcal{G}^w$ are consistent, then there are $z_i, i = 1, \ldots, n$ such that $|z_i|_X \neq 0$, $\forall\, i$ and $z_i - P_X(z_i) = w_{ij}(z_j - P_X(z_j))$, $\forall\,(i,j) \in \mathcal{E}^w$.

Proof: For any $(i_1, i_0) \in \mathcal{E}^w$ and $z_{i_0} \notin X$, take $e_{i_1 i_0} = w_{i_1 i_0}(z_{i_0} - P_X(z_{i_0}))$. Define a hyperplane

$$\mathcal{H}_\lambda = \{z | \langle z, e_{i_1 i_0}\rangle = \langle P_X(z_{i_0}) + \lambda e_{i_1 i_0}, e_{i_1 i_0}\rangle\}, \ \lambda \geq 0,$$

and denote the two corresponding closed half spaces as $\mathcal{H}_\lambda^+$ and $\mathcal{H}_\lambda^-$, respectively. Let

$$\lambda^* = \sup\{\lambda | \lambda \geq 0,\ X \bigcap \mathcal{H}_\lambda^+ \neq \emptyset\}.$$

Since $P_X(z_{i_0}) \in \mathcal{H}_0^+$ and $X \bigcap \mathcal{H}_\lambda^+ = \emptyset$ for sufficiently large λ, $\lambda^* < \infty$. It is easy to see that $X \bigcap \mathcal{H}_{\lambda^*}^+ \neq \emptyset$ and $X \subseteq \mathcal{H}_{\lambda^*}^-$. Let $z_{i_1} = y_{i_1} + e_{i_1 i_0}$ with $y_{i_1} \in X \bigcap \mathcal{H}_{\lambda^*}^+$. Clearly, $z_{i_1} \notin X$, $P_X(z_{i_1}) = y_{i_1}$ and $z_{i_1} - P_X(z_{i_1}) = e_{i_1 i_0} = w_{i_1 i_0}(z_{i_0} - P_X(z_{i_0}))$.

If $\mathcal{G}^w$ contains no weak cycle, we can apply the similar arguments to all the other arcs in $\mathcal{E}^w$ to obtain the conclusion; if $\mathcal{G}^w$ contains weak cycles and all its weak cycles are consistent, we can continue the above procedures until there are $z_1, \ldots, z_n$ such that $z_i - P_X(z_i) = w_{ij}(z_j - P_X(z_j))$ for all $(i,j) \in \mathcal{E}_*^w$, where $(\mathcal{V}, \mathcal{E}_*)^w$ is a maximal spanning subgraph of $\mathcal{G}^w$ containing no weak cycle. Because all weak cycles of $\mathcal{G}^w$ are consistent, $z_i - P_X(z_i) = w_{ij}(z_j - P_X(z_j))$ also holds for all the other w_{ij} when $(i,j) \in \mathcal{E}^w \backslash \mathcal{E}_*^w$. Thus, the proof is completed. ■

In the following two subsections, we will show that inconsistent cycles yield the inconsistent case for any initial conditions and the consistent cycles imply the consistent case for all initial conditions except a bounded set, respectively. Then in the third subsection, we will reveal the inherent relationships between consensus and our problem with the set containing only one point.

A. Set Surrounding

The following results provide sufficient conditions for the considered set surrounding problem.

Theorem 2: (i) The distributed set surrounding is achieved for system (1) with control law (2) if the communication graph $\mathcal{G}_\sigma$ is *UJSC* and all directed cycles of the configuration graph $\mathcal{G}^w$ are consistent; (ii) $\lim_{t\to\infty} |x_i(t)|_X = 0$, $i = 1, \ldots, n$ for any initial conditions if the communication graph $\mathcal{G}_{\sigma(t)} \equiv \mathcal{G}^w$ is fixed, strongly connected and there are inconsistent weak cycles in $\mathcal{G}^w$.

Proof: (i) Define the arc set connecting infinitely long time

$$\mathcal{E}_\infty = \left\{(i,j) | \exists\, \{t_k\}_{k=0}^\infty, t_k \to \infty \text{ such that } (i,j) \in \mathcal{E}_{\sigma(t_k)}\right\}$$

and corresponding graph $\mathcal{G}_\infty = (\mathcal{V}, \mathcal{E}_\infty)$. Clearly, $\mathcal{G}_\infty$ is a subgraph of the configuration graph $\mathcal{G}^w$.

Define

$$d(t) = \max_{1 \leq i \leq n} d_i(t), \qquad d_i(t) = \frac{1}{2}|x_i(t)|_X^2,\ i \in \mathcal{V},\ t \geq 0,$$

which are nonnegative. According to Proposition 1 in [15] (page 24), $|x_i(t)|_X^2$ is continuously differentiable and its derivative is $2\langle x_i^p(t), \dot{x}_i(t)\rangle$. Applying Lemma 1 gives

$$\begin{aligned} D^+ d(t) &= \max_{i \in \mathcal{I}(t)} \langle x_i^p(t), \dot{x}_i(t)\rangle \\ &= \max_{i \in \mathcal{I}(t)} \left\langle x_i^p(t), \sum_{j \in \mathcal{N}_i(\sigma(t))} \left(w_{ij} x_j^p(t) - x_i^p(t)\right)\right\rangle \\ &\leq \max_{i \in \mathcal{I}(t)} \sum_{j \in \mathcal{N}_i(\sigma(t))} \left(|x_i(t)|_X |x_j(t)|_X - |x_i(t)|_X^2\right) \leq 0 \end{aligned} \quad (3)$$

with $\mathcal{I}(t) = \{j | j \in \mathcal{V}, d_j(t) = d(t)\}$. Therefore, it follows from (3) that $d(t)$ is non-increasing and then converges to a finite number, that is,

$$\lim_{t\to\infty} d(t) = d^*. \quad (4)$$

As a result, the agent states $x_i(t), i \in \mathcal{V}, t \geq 0$ are bounded because X is bounded. In addition, if $d^* = 0$, the conclusion is obvious. Suppose $d^* > 0$ in the following proof of this part.

Since the switching communication graph $\mathcal{G}_\sigma$ is *UJSC*, by similar procedures in the proof of Lemma 4.3 in [19], we can show that $\lim_{t\to\infty} d_i(t) = d^*$, $i = 1, \ldots, n$. Because $\dot{d}_i$ is uniformly continuous, by Barbalat's Lemma (see Lemma 4.2 in [8]), we have $\lim_{t\to\infty} \dot{d}_i(t) = 0$, that is,

$$\begin{aligned} \lim_{t\to\infty} \dot{d}_i(t) &= \lim_{t\to\infty} \left\langle x_i^p(t), \sum_{j=1}^n \chi_{ij}(t)\left(w_{ij} x_j^p(t) - x_i^p(t)\right)\right\rangle \\ &= \lim_{t\to\infty} \sum_{j=1}^n \chi_{ij}(t)\left(-|x_i(t)|_X^2 + |x_j(t)|_X |x_i(t)|_X \right. \\ &\qquad \left. \times \cos\left(\angle w_{ij} x_j^p(t) - \angle x_i^p(t)\right)\right) \\ &= \lim_{t\to\infty} \sum_{j=1}^n \chi_{ij}(t)\left(-2d^* + 2d^* \cos\left(\angle w_{ij} x_j^p(t) - \angle x_i^p(t)\right)\right) \\ &= 0, \end{aligned}$$

which implies

$$\lim_{t\to\infty,\ t \in \Xi_{ij}} w_{ij} x_j^p(t) - x_i^p(t) = 0, \quad (5)$$

where $\Xi_{i,j} = \{t | (i,j) \in \mathcal{E}_{\sigma(t)}\}$,

$$\chi_{ij}(t) = \begin{cases} 1, & \text{if } (i,j) \in \mathcal{E}_{\sigma(t)} \\ 0, & \text{otherwise.} \end{cases}$$

It follows from (5) that for any $\varepsilon > 0$, there is $T_1 > 0$ such that, when $t \geq T_1$, $|w_{ij} x_j^p(t) - x_i^p(t)| \leq \varepsilon$ for $(i,j) \in \mathcal{E}_{\sigma(t)}$, and then $|\dot{x}_i(t)| \leq (n-1)\varepsilon$. Thus,$|x_i(t_2) - x_i(t_1)| \leq \int_{t_1}^{t_2} |\dot{x}_i(s)| ds \leq (n-1)(t_2 - t_1)\varepsilon$, and then $|x_i^p(t_2) - x_i^p(t_1)| \leq 2(n-1)(t_2 - t_1)\varepsilon$ for $t_2 \geq t_1 \geq T_1$, where the last inequality follows from the non-expansive property of projection operator: $|P_X(z_1) - P_X(z_2)| \leq |z_1 - z_2|, \forall z_1, z_2$. Without loss of generality, we assume T_1 is a sufficiently large number such that $\mathcal{E}_{\sigma(t)} \subseteq \mathcal{E}_\infty, \forall t \geq T_1$.

Take $(i_0, j_0) \in \mathcal{E}_\infty$ arbitrarily. Since the union graph $\mathcal{G}_\sigma([t, t+T))$ is strongly connected, there exist nodes $i_1, \ldots, i_k$, $k \leq n-2$ and time instants $t \leq s_0, s_1, \ldots, s_k < t + T$ such that $(i_r, i_{r+1}) \in \mathcal{E}_{\sigma(s_r)}$, $r = 0, \ldots, k-1$ and $(i_k, j_0) \in \mathcal{E}_{\sigma(s_k)}$. At the same time, there also exists a directed path $\mathcal{P}$ from j_0 to i_0 in $\mathcal{G}_\sigma([t, t+T))$. Denote the product of all configuration weights on $\mathcal{P}$ as w_*.

Since $\mathcal{G}_\sigma([t, t+T))$ is a subgraph of $\mathcal{G}^w$ and all directed cycles of $\mathcal{G}^w$ are consistent, all directed cycles of $\mathcal{G}_\sigma([t, t+T))$ are also consistent. Therefore, $\prod_{r=0}^{k-1} w_{i_r i_{r+1}} w_{i_k j_0} w_* = 1$. Moreover, since $i_0(i_0, j_0) j_0 \mathcal{P}$ is a directed cycle in $\mathcal{G}^w$, $w_{i_0 j_0} w_* = 1$. Thus, $\prod_{r=0}^{k-1} w_{i_r i_{r+1}} w_{i_k j_0} = w_{i_0 j_0}$ and

$$\begin{aligned} \left|x_{i_0}^p(t) - w_{i_0 j_0} x_{j_0}^p(t)\right| &= \left|x_{i_0}^p(t) - \prod_{r=0}^{k-1} w_{i_r i_{r+1}} w_{i_k j_0} x_{j_0}^p(t)\right| \\ &\leq \left|x_{i_0}^p(t) - x_{i_0}^p(s_0)\right| + \left|x_{i_0}^p(s_0) - w_{i_0 i_1} x_{i_1}^p(s_0)\right| \\ &\quad + \left|w_{i_0 i_1} x_{i_1}^p(s_0) - w_{i_0 i_1} x_{i_1}^p(s_1)\right| + \cdots \\ &\quad + \left|w_{k-2} x_{i_{k-1}}^p(s_{k-1}) - w_{k-1} x_{i_k}^p(s_{k-1})\right| \\ &\quad + \left|w_{k-1} x_{i_k}^p(s_{k-1}) - w_{k-1} x_{i_k}^p(s_k)\right| \\ &\quad + \left|w_{k-1} x_{i_k}^p(s_k) - w_{i_k j_0} w_{k-1} x_{j_0}^p(s_k)\right| \\ &\quad + \left|w_{i_k j_0} w_{k-1} x_{j_0}^p(s_k) - w_{i_k j_0} w_{k-1} x_{j_0}^p(t)\right| \\ &\leq (k+1)\varepsilon + 2(k+2)(n-1)^2 T\varepsilon \\ &\leq (n-1)\varepsilon + 2n(n-1)^2 T\varepsilon, \end{aligned}$$

where $w_q = \prod_{r=0}^{q} w_{i_r i_{r+1}}$. Since ε can be sufficiently small, we can further obtain

$$\lim_{t\to\infty} w_{ij}x_j^p(t) - x_i^p(t) = 0, \qquad \forall\,(i,j) \in \mathcal{E}_\infty. \tag{6}$$

Clearly, due to the uniformly strong connectivity of $\mathcal{G}_\sigma$, $\mathcal{G}_\infty$ is strongly connected. Combining the previous conclusion, (6) and the consistency of directed cycles of $\mathcal{G}^w$, we have $\lim_{t\to\infty} w_{ij}x_j^p(t) - x_i^p(t) = 0$ for all the other w_{ij} with $(i,j) \in \mathcal{E}^w \backslash \mathcal{E}_\infty$. Thus, the proof of part (i) is completed.

(ii) We first show this conclusion for the case when there exist inconsistent directed cycles in $\mathcal{G}^w$. Let $i_1 e_1 i_2 e_2 \cdots i_k e_k i_1$ be an inconsistent directed cycle in $\mathcal{G}^w$ with

$$\prod_{r=1}^{k} w_{i_r i_{r+1}} \neq 1, \tag{7}$$

$i_{k+1} = i_1$. From (5), we have $\lim_{t\to\infty} w_{i_r i_{r+1}} x_{i_{r+1}}^p(t) - x_{i_r}^p(t) = 0$ for $1 \le r \le k$. Therefore,

$$\lim_{t\to\infty} x_{i_1}^p(t)\left(1 - \prod_{r=1}^{k} w_{i_r i_{r+1}}\right) = 0,$$

which implies $\lim_{t\to\infty} d_{i_1}(t) = 0$ and then $d^* = 0$. For the case of existing weak cycles (not directed cycles) in $\mathcal{G}^w$, we can similarly show this conclusion by replacing the configuration weight $w_{i_r i_{r+1}}$ in (7) with $w_{i_{r+1} i_r}^{-1}$ corresponding to arc $e_r = (i_{r+1}, i_r)$.

Thus, we complete the proof. ■

From the proof of Theorem 2, we can find that the conclusion (ii) also holds under the following relaxed connectivity condition: the communication graph $\mathcal{G}_\sigma$ is *UJSC* and there exist a time sequence $\{s_k\}_{k=0}^\infty$, $s_k \to \infty$ and $b_0 > 0$ such that $\mathcal{E}^w \subseteq \bigcup_{t=s_k}^{s_k+b_0} \mathcal{E}_{\sigma(t)}$ for $k \ge 0$.

B. Consistent Case

Theorem 2 showed that the distributed set surrounding can be achieved under *UJSC* communication graph condition. In this subsection, we further show under the case without inconsistent cycles of $\mathcal{G}^w$, how to select the initial conditions such that the consistent case (that is, $d^* > 0$ given in (4) can be guaranteed.

Let $L_{\sigma(t)}$ be the matrix with entries

$$\left(L_{\sigma(t)}\right)_{ij} = \begin{cases} |\mathcal{N}_i(\sigma(t))|, & \text{if } i = j \\ -w_{ij}, & \text{if } i \neq j, j \in \mathcal{N}_i(\sigma(t)) \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Then system (1) with control law (2) can be written in the following compact form:

$$\dot{x}(t) = -L_{\sigma(t)} x^p(t), \tag{9}$$

where $x(t) = (x_1(t), \dots, x_n(t))^T$ is the stack vector of agents' states, $x^p(t) = (x_1^p(t), \dots, x_n^p(t))^T$ is the stack vector of agents' projection vectors.

We first consider system (9) with a *UJSC* undirected graph $\mathcal{G}_\sigma$, where all directed cycles of the configuration graph $\mathcal{G}^w$ are consistent. Without loss of generality (otherwise we can relabel the index of nodes), we take a spanning tree $\mathcal{T}$ of $\mathcal{G}^w$ as follows: $\mathcal{T} = \bigcup_{k=1}^{\rho} \mathcal{T}_k$, where the initial and the terminal nodes of the path $\mathcal{T}_k$ are i_k and 1, respectively; the nodes in the path from i_k to 1 are in the order $i_k, i_k - 1, \dots, i_{k-1}+1, 1, 1 \le k \le \rho, i_0 = 1$. Associated with the n nodes, we define n nonzero complex numbers

$$p_1 = 1,\ p_j = w_{(i_{k-1}+1)1}^{-1} \prod_{r=i_{k-1}+1}^{j-1} w_{(r+1)r}^{-1} \text{ for } i_{k-1}+1 \le j \le i_k.$$

Denote a diagonal matrix

$$P = \mathrm{diag}(p_1, \dots, p_n) \tag{10}$$

with diagonal elements $p_i, 1 \le i \le n$. It is easy to see that $\widetilde{L}_{\sigma(t)} = P L_{\sigma(t)} P^{-1}$ is the Laplacian[1] of undirected graph $\mathcal{G}_{\sigma(t)}$. Then we have

Theorem 3: In the switching *UJSC* undirected graph case, $d^* > 0$ if the initial condition $x(0)$ satisfies $|(\mathbf{1}^T P x(0))/n| > \sup_{z\in X} |z|$.

Proof: Let $\widetilde{x}(t) = Px(t)$. Clearly, system (9) can be written as

$$\dot{\widetilde{x}}(t) = -\widetilde{L}_{\sigma(t)} P x^p(t).$$

Because $\mathbf{1}^T \widetilde{L}_{\sigma(t)} \equiv 0$ with $\mathbf{1} = (1, \dots, 1)^T$, $\mathbf{1}^T \widetilde{x}(t)/n$ is time-invariant. Note that $\sup_{z\in X} |z|$ is a finite number since X is bounded.

We prove the conclusion by contradiction. Hence suppose $d^* = 0$. Since $\lim_{t\to\infty} d_i(t) = d^*$ under the *UJSC* assumption, $\lim_{t\to\infty} |x_i(t)|_X = 0$. Therefore, $\limsup_{t\to\infty} |x_i(t)| \le \sup_{z\in X} |z|$ and then $|\mathbf{1}^T \widetilde{x}(t)/n| \le \sup_{z\in X} |z|$, which yields a contradiction due to $\mathbf{1}^T \widetilde{x}(t)/n \equiv \mathbf{1}^T \widetilde{x}(0)/n$. ■

Next we consider system (9) under a fixed strongly connected digraph $\mathcal{G}_{\sigma(t)} \equiv \mathcal{G}^w$ with all its directed cycles being consistent. Since any strongly connected graph contains a directed spanning tree, $\mathcal{G}^w$ contains a directed spanning tree $\mathcal{T}^d = \bigcup_{k=1}^{\varrho} \mathcal{T}_k^d$, where the initial and the terminal node of the directed path $\mathcal{T}_k^d$ are i_k and 1, respectively. Moreover, the nodes in the directed path from i_k to 1 are in the order $i_k, i_k - 1, \dots, i_{k-1}+1, 1, 1 \le k \le \varrho, i_0 = 1$. Associated with the n nodes, we can similarly define

$$q_1 = 1,\ q_j = w_{(i_{k-1}+1)1}^{-1} \prod_{r=i_{k-1}+1}^{j-1} w_{(r+1)r}^{-1} \text{ for } i_{k-1}+1 \le j \le i_k.$$

Let $Q = \mathrm{diag}(q_1, \dots, q_n)$. It is easy to see that $QL_{\sigma(t)}Q^{-1} \equiv QL_{\sigma(0)}Q^{-1}$ is the Laplacian of the fixed digraph $\mathcal{G}^w$ and

$$\alpha^T Q x(t) \tag{11}$$

is time-invariant, where $\alpha = (\alpha_1, \dots, \alpha_n)^T$ with $\alpha_i > 0$, $\sum_{i=1}^n \alpha_i = 1$ is the left eigenvector of QLQ^{-1} associated with eigenvalue 0, that is, $\alpha^T QLQ^{-1} = 0$. Similar to the undirected graph case, we can show the following result, whose proof is omitted due to space limitations.

Theorem 4: In the fixed strongly-connected digraph case, $d^* > 0$ if the initial condition $x(0)$ satisfies $|\alpha^T Q x(0)| > \sup_{z\in X} |z|$.

Remark 4: Clearly, by the relation (3) we always have $d^* \le \max_{1\le i\le n} |x_i(0)|_X$. Generally, the final distance d^* between agents and X depends on the initial conditions, graph $\mathcal{G}_\sigma$, matrix W and the shape of X. The computation of d^* is very complicated and it is not easy to give its value, or even a lower bound because our connectivity condition and convex set are quite general. On the other hand, in some special cases, we can certainly discuss d^*. For example, when $\mathcal{G}_\sigma$ is undirected, *UJSC* and X is a ball with center $(0,0)$ and radius r_0, if $|(\mathbf{1}^T \widetilde{x}(0))/n| > r_0$ (the sufficient condition in Theorem 3 is satisfied), then

$$d^* \ge \sqrt{2\left(\left|\frac{\mathbf{1}^T \widetilde{x}(0)}{n}\right| - r_0\right)} > 0$$

because $\lim_{t\to\infty}(|x_i(t)|_X - |x_j(t)|_X) = 0$ with $|x_i(t)|_X = |\widetilde{x}_i(t)|_X$ and $\lim_{t\to\infty} |\widetilde{x}_i(t)|_X \ge |\mathbf{1}^T \widetilde{x}(t)/n|_X = |\mathbf{1}^T \widetilde{x}(t)/n| - r_0$ with $\mathbf{1}^T \widetilde{x}(t)/n \equiv \mathbf{1}^T \widetilde{x}(0)/n$. Similar estimation can also be given for the fixed strongly-connected digraph case.

[1]The Laplacian $\bar{L}$ of a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined as: $(\bar{L})_{ii} = |\bar{\mathcal{N}}_i|$, $(\bar{L})_{ij} = -1$ for $j \neq i, j \in \bar{\mathcal{N}}_i$ and all other entries are zero, where $\bar{\mathcal{N}}_i = \{j | (i,j) \in \mathcal{E}\}$ [18].

C. Special Case: $X = \{(0,0)\}$

Here we consider a special case when the set becomes a point. Without loss of generality, take $X = \{(0,0)\}$, which can be regarded as a stationary leader of the multi-agent system. Then system (1) with control law (2) can be rewritten as

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i(\sigma(t))} (w_{ij}x_j(t) - x_i(t)) \tag{12}$$

or in the compact form: $\dot{x}(t) = -L_{\sigma(t)}x(t)$, where L_σ is given in (8).

Remark 5: System (12) is a generalized model for various models in the multi-agent literature. For example, when $w_{ij} = 1$ for $(i,j) \in \mathcal{E}^w$, system (12) becomes the standard consensus model with all connection weights equal to 1. Moreover, the bipartite consensus model discussed in [13] is a special case of system (12) with $w_{ij} = 1$ or -1.

Remark 6: Different from the feedback control $u_i = \sum_{j\in\mathcal{N}_i} w_{ij}(x_j - x_i)$ given in [20], [21], our distributed control is $u_i = \sum_{j\in\mathcal{N}_i}(w_{ij}x_j - x_i)$. As stated in [21], the system matrix generated by v_i may have eigenvalues with positive real parts and then the resulting system may be unstable. Here if the graph $\mathcal{G}_\sigma$ is undirected and switching (or fixed and strongly connected) with consistent directed cycles of $\mathcal{G}^w$, then all the eigenvalues of L_σ (or L) have non-negative real parts, which implies that for the two cases system (12) is always stable.

Consider system (12) associated with a *UJSC* undirected graph $\mathcal{G}_\sigma$. Recalling the diagonal matrix P in (10), we first have the following theorem.

Theorem 5: For system (12), if the undirected communication graph $\mathcal{G}_\sigma$ is *UJSC* and all directed cycles of the configuration graph $\mathcal{G}^w$ are consistent, then, for any initial condition $x_i(0), i = 1,\ldots,n$,

$$\lim_{t\to\infty} x_i(t) = \frac{\sum_{j=1}^n p_j x_j(0)}{np_i}, \qquad 1 \le i \le n.$$

Proof: Recalling the notations $\widetilde{x}(t)$ and $\widetilde{L}_{\sigma(t)}$ used in Section III-B, we have $\dot{\widetilde{x}}(t) = -\widetilde{L}_{\sigma(t)}\widetilde{x}(t)$. According to Theorem 2.33 in [3], for any $x_i(0),\ 1 \le i \le n$,

$$\lim_{t\to\infty} \widetilde{x}_i(t) = \frac{\sum_{j=1}^n \widetilde{x}_j(0)}{n}, \qquad 1 \le i \le n,$$

which implies the conclusion. ■

Next we consider system (12) with a fixed strongly connected digraph $\mathcal{G}_{\sigma(t)} \equiv \mathcal{G}^w$. Clearly, L is diagonally dominant and all its eigenvalues are either 0 or with positive real parts.

Lemma 2: 0 is an eigenvalue of L if and only if all directed cycles of $\mathcal{G}^w$ are consistent.

Proof: Sufficiency. If all directed cycles of $\mathcal{G}^w$ are consistent, then by the discussions in SubSection III-B, there is an invertible diagonal matrix Q such that QLQ^{-1} is the Laplacian of digraph $\mathcal{G}^w$. Since all row sums of any Laplacian are zero, any Laplacian has an eigenvalue zero. The sufficiency follows from that similar matrices have the same eigenvalues.

Necessity. Let us show it by contradiction. Suppose that $\mathcal{G}^w$ contains inconsistent directed cycles. On one hand, by Theorem 2 (ii), $\lim_{t\to\infty} x_i(t) = 0,\ i = 1,\ldots,n$ for any initial conditions (noticing that $X = \{(0,0)\}$). On the other hand, let $\xi \ne 0$ be the eigenvector of L with eigenvalue 0, that is, $L\xi = 0$. Clearly, $x(t) \equiv \xi$ for initial condition $x(0) = \xi$, which yields a contradiction. Thus, the necessity follows. ■

Recalling the matrix Q and vector α defined in (11) along with Lemma 2 and Theorem 2.13 in [3], we have the following theorem.

Theorem 6: Consider system (12) with a fixed strongly connected digraph $\mathcal{G}_{\sigma(t)} \equiv \mathcal{G}^w$. Then $\lim_{t\to\infty} x_i(t) = 0,\ i = 1,\ldots,n$ for any initial conditions if and only if $\mathcal{G}^w$ contains inconsistent directed cycles. Moreover, if all directed cycles of $\mathcal{G}^w$ are consistent, then, for any initial condition $x_i(0), i = 1,\ldots,n$,

$$\lim_{t\to\infty} x_i(t) = \frac{\sum_{j=1}^n \alpha_j q_j x_j(0)}{q_i}, \qquad 1 \le i \le n.$$

Remark 7: Clearly, the results in Theorems 5 and 6 are consistent with the conventional results in [2], [3], [13]. In fact, if all w_{ij}'s are 1, both P and Q given in SubSection III-B are the identity matrix, which implies that all agents will achieve a consensus for any initial conditions by the conclusions in Theorems 5 and 6. Moreover, Theorem 2 in [13] showed that all agents will converge to the origin for the structurally unbalanced graph case or achieve the bipartite consensus for the structural balanced graph case, which can be obtained from Theorem 6 in this technical note by noticing that a digraph with all configuration weights being -1 or 1 is structurally balanced if and only if all its directed cycles are consistent. Due to the convex set and complex-value weights, the method given in [13] for the bipartite consensus cannot be applied directly to solve our problem.

Sometimes, we need to check whether the weak cycles in the configuration graph are consistent and it is known that the consistency of weak cycles in digraphs implies that of directed cycles. In the strongly connected digraph case, the converse is also true and then we only need to check the consistency of all directed cycles instead of that of all weak cycles as the next result shows.

Theorem 7: Suppose $\mathcal{G}^w$ is strongly connected. Then all directed cycles of $\mathcal{G}^w$ are consistent if and only if all weak cycles of $\mathcal{G}^w$ are consistent.

Proof: The sufficiency is straightforward. We focus on the necessity. Without loss of generality, let the weak cycle in $\mathcal{G}^w$ take the following form:

$$i_1 e_1 i_2 e_2 \cdots e_{k_1-1} i_{k_1} e_{k_1}^{-1} i_{k_1+1} \cdots i_k e_k^{-1} i_1,$$

where $e_r = (i_r, i_{r+1}),\ r = 1,\ldots,k_1-1;\ e_r^{-1} = (i_{r+1}, i_r),\ r = k_1,\ldots,k, i_{k+1} = i_1$. Since $\mathcal{G}^w$ is strongly connected, for each $r = k_1,\ldots,k$, there is a directed path $\mathcal{P}_r$ from i_r to i_{r+1}. Because the directed cycles $\mathcal{P}_r e_r^{-1}, r = k_1,\ldots,k$ are consistent, $w(\mathcal{P}_r)w_{i_{r+1}i_r} = 1$, where $w(\mathcal{P}_r)$ is the product of all configuration weights on directed path $\mathcal{P}_r$. Then from the consistency of the directed cycle $i_1 e_1 i_2 e_2 \cdots e_{k_1-1}\mathcal{P}_{k_1}\cdots\mathcal{P}_k$, we have

$$\begin{aligned} &w_{i_1i_2}\cdots w_{i_{k_1-1}i_{k_1}} w^{-1}_{i_{k_1+1}i_{k_1}} \cdots w^{-1}_{i_{k+1}i_k} \\ &\qquad = w_{i_1i_2}\cdots w_{i_{k_1-1}i_{k_1}} w(\mathcal{P}_{k_1})\cdots w(\mathcal{P}_k) = 1. \end{aligned}$$

Thus, the conclusion follows. ■

IV. Numerical Example

In this section, we provide an example to illustrate the results obtained in this technical note.

Consider a network of five agents with node set $\mathcal{V} = \{1,2,3,4,5\}$ and the complex-value adjacency matrix $W = (w_{ij})$. The convex set to be surrounded is the unit ball in $\mathbb{R}^2$. The initial conditions are $x_1(0) = 2 + 4\iota,\ x_2(0) = 4 + 3\iota,\ x_3(0) = -4 - 3\iota,\ x_4(0) = -4 + 2\iota, x_5(0) = 2 + 3\iota$ (marked as $\circ$ in Figs. 1 and 2).

- Consistent case: Take $w_{12} = w_{23} = w_{34} = e^{(\pi/2)\iota},\ w_{45} = e^{(\pi/3)\iota},\ w_{51} = e^{(\pi/6)\iota}$, and all other configuration weights are zero. Then the resulting configuration graph is $\mathcal{G}^w = (\mathcal{V}, \mathcal{E}^w)$ with arc set $\mathcal{E}^w = \{(1,2),(2,3),(3,4),(4,5),(5,1)\}$.

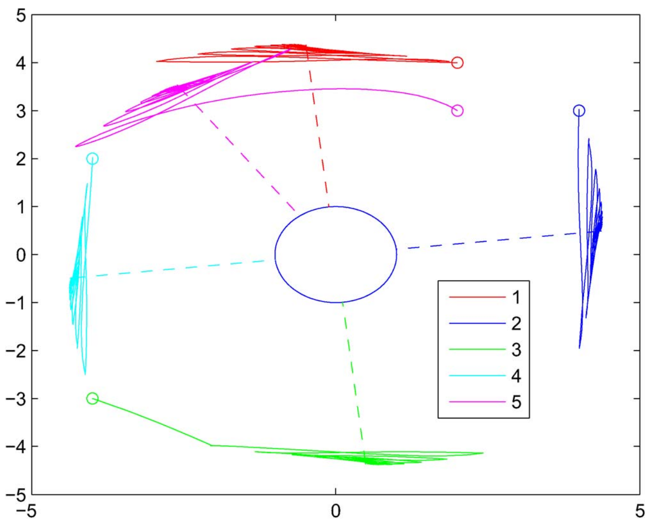

Fig. 1. The consistent cycles yield the consistent set surrounding.

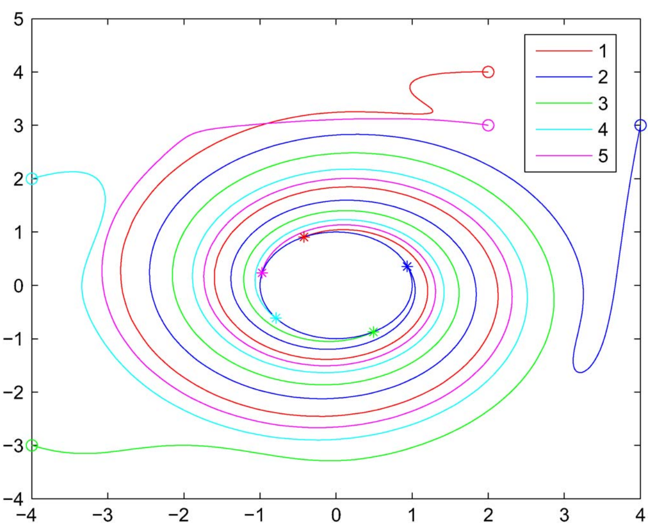

Fig. 2. The inconsistent cycles yield the inconsistent set surrounding.

The communication graph of the multi-agent system is periodically switched between two graphs $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1), \mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$ with $\mathcal{E}_1 = \{(1,2),(3,4),(5,1)\}$ and $\mathcal{E}_2 = \{(2,3),(4,5)\}$ in the following order: $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_2, \ldots$ with switching period 5. Clearly, $\mathcal{G}_\sigma$ is *UJSC* and all directed cycles of $\mathcal{G}^w$ are consistent. Fig. 1 demonstrates that all agents accomplish the consistent set surrounding at time $t = 2000$, where the five agent trajectories are described by the solid lines and the projection vectors of the final positions of the agents are described by dashed lines.

- Inconsistent case: Take $w_{12} = w_{23} = w_{34} = e^{(\pi/2)\iota}$, $w_{45} = e^{(\pi/3)\iota}$, $w_{51} = e^{(\pi/3)\iota}$, $w_{14} = e^{(\pi/2)\iota}$, and all other configuration weights are zero. Suppose the communication graph is fixed, that is, $\mathcal{G}_\sigma \equiv \mathcal{G}^w = (\mathcal{V}, \mathcal{E}^w)$ with $\mathcal{E}^w = \{(1,2),(2,3),(3,4),(4,5),(5,1),(1,4)\}$. Note that the configuration graph $\mathcal{G}^w$ defined by the new configuration weights is clearly inconsistent. Fig. 2 shows that all agents converge to the unit ball, where the final positions of the five agents at time $t = 2000$ are marked with *.

V. Conclusion

In this technical note, we proposed a formulation and a distributed controller for set surrounding problems. We discussed both consistent and inconsistent cases, and obtained the necessary/sufficient conditions for multi-agent systems with communication topologies described by joint-connected graphs. Moreover, we showed when the consistent case can be guaranteed, and also provided conditions on the leader-following consensus when the target set becomes one point.

References

[1] M. Earl and R. D'Andrea, "A study in cooperative control: The Roboflag drill," in *Proc. Amer. Control Conf.*, Anchorage, AK, May 2002, pp. 1811–1812.

[2] R. Olfati-Saber, J. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[3] W. Ren and R. Beard, *Distributed Consensus in Multi-Vehicle Cooperative Control*. London, U.K.: Springer-Verlag, 2008.

[4] J. Hu and G. Feng, "Distributed tracking control of leader-follower multiagent systems under noisy measurement," *Automatica*, vol. 46, no. 8, pp. 1382–1381, 2010.

[5] Z. Chen and H. Zhang, "No-beacon collective circular motion of jointly connected multi-agents," *Automatica*, vol. 47, no. 9, pp. 1929–1937, 2011.

[6] B. D. O. Anderson, C. Yu, S. Dasguptab, and A. Morse, "Control of a three-coleader formation in the plane," *Syst. Control Lett.*, vol. 56, no. 9–10, pp. 573–578, 2007.

[7] P. Lin, K. Qin, Z. Li, and W. Ren, "Collective rotating motions of secondorder multi-agent systems in three-dimensional space," *Syst. Control Lett.*, vol. 60, no. 6, pp. 365–372, 2011.

[8] J. E. Slotine and W. Li, *Applied Nonlinear Control*. Englewood Cliffs, NJ, USA: Prentice-Hall, 2000.

[9] G. Shi and Y. Hong, "Global target aggregation and state agreement of nonlinear multi-agent systems with switching topologies," *Automatica*, vol. 45, no. 5, pp. 1165–1175, 2009.

[10] G. Shi, Y. Hong, and K. Johansson, "Connectivity and set tracking of multi-agent systems guided by multiple moving leaders," *IEEE Trans. Autom. Control*, vol. 57, no. 3, pp. 663–676, 2012.

[11] Y. Lou and Y. Hong, "Target containment control of multi-agent systems with random switching interconnection topologies," *Automatica*, vol. 48, no. 5, pp. 879–885, 2012.

[12] Y. Cao, D. Stuart, W. Ren, and Z. Meng, "Distributed containment control for multiple autonomous vehicles with double-integrator dynamics: Algorithms and experiments," *IEEE Trans. Control Syst. Technol.*, vol. 19, no. 4, pp. 929–938, 2010.

[13] C. Altafini, "Consensus problems on networks with antagonistic interactions," *IEEE Trans. Autom. Control*, vol. 58, no. 4, pp. 935–946, 2013.

[14] R. Rockafellar, *Convex Analysis*. Princeton, NJ, USA: Princeton Univ. Press, 1972.

[15] J. Aubin and A. Cellina, *Differential Inclusions*. Berlin, Germany: Springer-Verlag, 1984.

[16] J. Danskin, "The theory of max-min, with applications," *SIAM J. Appl. Math.*, vol. 14, no. 4, pp. 641–664, 1966.

[17] F. Chen, W. Ren, and Y. Cao, "Surrounding control in cooperative agent networks," *Syst. Control Lett.*, vol. 59, no. 11, pp. 704–712, 2010.

[18] C. Godsil and G. Royle, *Algebraic Graph Theory*. New York, NY, USA: Springer-Verlag, 2001.

[19] G. Shi, K. Johansson, and Y. Hong, "Reaching an optimal consensus: Dynamical systems that compute intersections of convex sets," *IEEE Trans. Autom. Control*, vol. 58, no. 3, pp. 610–622, 2013.

[20] W. Ding, G. Yan, and Z. Lin, "Collective motions and formations under pursuit strategies on directed acyclic graphs," *Automatica*, vol. 46, no. 1, pp. 174–181, 2010.

[21] L. Wang, Z. Han, Z. Lin, and G. Yan, "Complex Laplacian and pattern formation in multi-agent systems," in *Proc. Chinese Control Decision Conf.*, Taiyuan, China, May 2012, pp. 628–633.

[22] W. Ren, "Collective motion from consensus with cartesian coordinate coupling," *IEEE Trans. Autom. Control*, vol. 54, no. 6, pp. 1330–1335, 2009.