Journal of Mathematical Economics 90 (2020) 119–126

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

A new approach to the existence and regularity of linear equilibrium in a noisy rational expectations economy[☆]

Youcheng Lou [*], Shouyang Wang

MDIS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, China

ARTICLE INFO

Article history:
Received 2 January 2020
Received in revised form 3 June 2020
Accepted 8 July 2020
Available online 24 July 2020

Keywords:
Existence and regularity
Rational expectations equilibrium
A new approach

ABSTRACT

This paper gives a new approach to show the existence and regularity of linear equilibrium established by Lou ⓡ al. (2019) for a noisy rational expectations economy. Different from the existing method which essentially requires to find a fixed point of a system of nonlinear algebraic equations, the new approach is operated directly on an alternative form of market-clearing conditions. One main advantage of the new approach is that besides homogeneous-valuation economies, it can also handle the existence of equilibrium in economies with heterogeneous valuations where the existing method for dealing with homogeneous-valuation economies fails to work.

1. Introduction

Rational expectations equilibrium (REE) economies have been widely studied since the pioneering works of Grossman (1976), Hellwig (1980) and Grossman and Stiglitz (1980). Grossman (1976) proposes a REE economy with a constant supply of the risky asset and show that the equilibrium price perfectly aggregates all private information of market participants. To prevent prices from becoming fully revealing, noise trading is introduced into the economy to make the equilibrium price only partially revealing in a finite-agent setting (Hellwig, 1980) and a continuum-agent setting (Grossman and Stiglitz, 1980).

Lou ⓡ al. (2019) generalize the finite-agent economy in Hellwig (1980) to a continuum-agent economy with general signal structure where traders' signals are multidimensional and arbitrary correlation pattern between the components of traders' signals and between the fundamental of the risky asset and signals is allowed.[1] Lou ⓡ al. (2019) transfer equivalently the equilibrium existence problem into a fixed-point existence problem. When there is no idiosyncratic noise, the authors develop a new technique to solve the equilibrium existence problem because in this case, the underlying fixed-point function is not uniformly bounded and consequently, Brouwer's fixed-point theorem cannot be applied directly. Specifically, the authors first construct an auxiliary sequence of uniformly bounded functions, and then get a fixed point for every such a function. Taking the limit point of this sequence of fixed points (which is shown to be bounded) gets a fixed point of the original function which involves a system of nonlinear algebraic equations, coming from coefficient matching based on market-clearing conditions.[2]

This paper aims to present a *new* approach to show the *existence* and *regularity* of linear equilibrium in the noisy REE economy in Lou ⓡ al. (2019). The existence of linear equilibrium is shown by applying Brouwer's fixed-point theorem to a function constructed directly from an alternative form of (one-dimensional) market-clearing conditions, *no longer* to a system of nonlinear algebraic equations (which was done in Lou ⓡ al. (2019)). To be specific, we first construct an auxiliary price function in which an independent random variable is additionally introduced, and write the market-clearing condition in an alternative form where one side is the price function, while the other side is a term involving a variance-adjusted conditional expectation. We then construct two functions which map to the coefficients on signals and noise trade in the price function based on the alternative form of market-clearing conditions, and show the continuity and uniform boundedness of the two functions based on some elegant properties of conditional expectation and variance. Applying Brouwer's fixed-point theorem to get a

☆ This work was supported in part by the National Natural Science Foundation of China under Grant 71971208.

* Corresponding author at: Academy of Mathematics and Systems Science, Chinese Academy of Sciences, No. 55 Zhongguancun East Road, Beijing 100190, China.

E-mail addresses: louyoucheng@amss.ac.cn (Y. Lou), sywang@amss.ac.cn (S. Wang).

[1] The generality of dimension and correlation pattern of signals can help formulate the situation where traders share signals with their neighbors in a social network. See Lou ⓡ al. (2019) for more potential applications of the generality.

[2] When there is idiosyncratic noise, the proof of existence of linear equilibrium is easy by applying directly Brouwer's fixed-point theorem because in this case the underlying fixed-point function is uniformly bounded and continuous, see the proof of Proposition 1 in Lou ⓡ al. (2019) for more details.

https://doi.org/10.1016/j.jmateco.2020.07.003
0304-4068/

sequence of fixed-points. In addition, we show that any limit point of the coefficient on noise trade does not equal zero. Finally, taking the limit of both sides of the alternative market-clearing condition gives a linear equilibrium. Furthermore, in two cases when noise trading is large or signals take the classical form of the sum of the fundamental and an independent noise, the regularity of equilibrium prices (i.e., an increase in noise demand implies a higher price) is also established.

Working directly on the one-dimensional market-clearing condition (instead of a system of nonlinear equations) facilitates us to utilize some elegant properties of conditional expectation and variance, for example, the boundedness and monotonicity of conditional variance with respect to traders' signals, the law of total variance, etc. It is worth remarking that our new approach can also be applied to show the existence and regularity of an extended version of the model in Lou ⓡ al. (2019) where traders valuate the risky asset *heterogeneously*. However, the method in Lou ⓡ al. (2019) *cannot* be applied to solve equilibrium existence of economies with heterogeneous valuations because it depends crucially on one property, which does not hold for heterogeneous-valuation economies in general; see Section 4 for more illustrations.

Except for the work mentioned above, our work also relates to the literature on equilibrium existence and uniqueness of REE economies. Pálvölgyi and Venter (2015a) show that the linear equilibrium in Grossman and Stiglitz (1980) is unique in the class of all continuous price functions. Barlevy and Veronesi (2000) consider a setting where the fundamental is binomial and investors are risk-neutral instead of the classical normality-CARA assumption. Breon-Drish (2015) analyzes the finite-agent economy of Hellwig (1980), but with an extension to more general signal structures of the exponential family. There is also literature that studies existence and uniqueness of an equilibrium for economies with multiple risky assets, for instance, Pálvölgyi and Venter (2015b), Chabakauri et al. (2017) and Carpio and Guo (2019). In addition, our work is also related to heterogeneous-valuation economies; see Rostek and Weretka (2012), Vives (2014) and Rahi and Zigrand (2018). Indeed, traders are possibly uncertain about the values of risky assets and valuate them heterogeneously. Heterogeneity of valuations can also be interpreted as arising from the fact that traders valuate their investment from different perspectives, for instance, short-term returns, long-term returns, and the volatility of prices, etc.

The paper is organized as follows: Section 2 introduces the model. Section 3 presents the new approach to show the existence of linear equilibrium, and regularity of linear equilibrium in two special cases. Section 4 offers an extension to economies with heterogeneous valuations. Section 5 concludes the paper. All preliminary lemmas are in the Appendix.

Details on notation. We follow the notation in Lou ⓡ al. (2019). All vectors are column vectors by default. The operator Var will stand for variance and Cov will stand for covariance. For any vector $\boldsymbol{\mu} = (\boldsymbol{\mu}_1', \ldots, \boldsymbol{\mu}_n')'$ (where each component $\boldsymbol{\mu}_i$ is m-dimensional, and $'$ denotes the transpose of a vector) and random variable x, $Cov(\boldsymbol{\mu}, x)$ is shorthand for $Cov(\sum_{k=1}^n \boldsymbol{\mu}_k' \boldsymbol{y}_k, x)$ and $\mathrm{Var}(\boldsymbol{\mu})$ stands for $\mathrm{Var}(\sum_{k=1}^n \boldsymbol{\mu}_k' \boldsymbol{y}_k)$, where $\boldsymbol{y} = (\boldsymbol{y}_1', \ldots, \boldsymbol{y}_n')'$ is the aggregate signal in the economy. For any random variable x and m-dimensional signal $\boldsymbol{y}_i = (y_{i1}, \ldots, y_{im})'$, $\mathbf{Cov}(x, \boldsymbol{y}_i)$ is shorthand for the vector of covariances $(Cov(x, y_{i1}), \ldots, Cov(x, y_{im}))'$. For any multidimensional random vector $\boldsymbol{z}$, $\mathbf{Var}(\boldsymbol{z})$ denotes the variance–covariance matrix of $\boldsymbol{z}$. Finally, $|\cdot|$ denotes the 2-norm of a vector.

2. The model

Consider the economy in Lou ⓡ al. (2019) with a specification that there is no idiosyncratic noise.[3] There is a single risky asset and a single trading period. The risky asset is in fixed supply, which is assumed to be zero for simplicity, and has fundamental value θ, which is common to all traders, but not directly observed by traders.

There are n traders in the economy. Each trader i ($i = 1, \ldots, n$) has a CARA utility function and maximizes her conditional expected utility of her net profit W_i based on her information set $\mathcal{F}_i$:

$$\mathbb{E}\left[-\exp\{-\rho_i W_i\} | \mathcal{F}_i\right].$$

Here ρ_i represents trader i's coefficient of absolute risk aversion, $W_i = x_i(\theta - p)$ with x_i being the holdings of the risky asset and p its price. Due to the CARA assumption, each trader's initial wealth is without loss of generality assumed to be zero here.

Besides the price p, each trader i can also observe a multidimensional private signal $\boldsymbol{y}_i \in \mathbb{R}^m$. That is, the information set of trader i is given by

$$\mathcal{F}_i = \{\boldsymbol{y}_i, p\}.$$

We use $\boldsymbol{y} = (\boldsymbol{y}_1', \ldots, \boldsymbol{y}_n')'$ to denote the *aggregate signal* in the economy. The private signal $\boldsymbol{y}_i$ and the price p together determine trader i's demand; see (1) below. In addition to such demands, there is also noise demand u in the economy, which is interpreted as the stochastic demand of "noise traders" left unmodeled. We impose the following assumption, which will be maintained throughout.

Assumption 1. All random variables are normally distributed, with means normalized to zero. The variance–covariance matrix $\mathbf{Var}(\boldsymbol{y}_i)$ is positive definite for every i, and $\mathrm{Var}(\theta|\boldsymbol{y}) > 0$. Noise demand u is independent of other random variables in the model, and has a positive variance.

Except for joint normality and the requirement that all signals in the economy (i.e., the aggregate signal $\boldsymbol{y}$) cannot fully pin down the fundamental, very *little* restriction is imposed on the correlation pattern between the components of $\boldsymbol{y}$ or on the precise relationship of its components with the fundamental θ. Due to the generality of dimension and correlation pattern of signals, several existing economies serve as special cases; see for example, Grossman and Stiglitz (1980), Grossman (1976), Hellwig (1980) and Ozsoylev and Walden (2011). Importantly, we do not require that the variance–covariance matrix of the aggregate signal $\boldsymbol{y}$ is positive definite. This weak requirement permits the considered economy to deal with the situation that traders share signals with their neighbors in a social network; see Lou ⓡ al. (2019) for more illustrations on the generality of the considered economy.

Under the CARA-normality setting, it is well-known that the optimal demand for the risky asset by trader i is given by[4]

$$x_i^* = \frac{\mathbb{E}(\theta|\boldsymbol{y}_i, p) - p}{\rho_i \mathrm{Var}(\theta|\boldsymbol{y}_i, p)}. \tag{1}$$

[3] Note that when there is no idiosyncratic noise, the continuum-agent economy in Lou ⓡ al. (2019) is mathematically equivalent to the finite-agent economy considered here. Furthermore, the economy considered in this paper also coincides with the economy in Hellwig (1980) except for an extension to permit multidimensional signals and arbitrary signal structure (still remaining within the standard framework of normality).

[4] Notice that the equality (1) is well defined because $\mathrm{Var}(\theta|\boldsymbol{y}_i, p) \geq \mathrm{Var}(\theta|\boldsymbol{y}, u) = \mathrm{Var}(\theta|\boldsymbol{y}) > 0$ by Assumption 1 and Lemma 4 in the Appendix.

A *rational expectations equilibrium* is defined as a price p, together with the optimal demands x_i^*'s, which are given by (1), satisfy the following market-clearing condition:

$$\sum_{i=1}^{n} x_i^* + u = 0. \tag{2}$$

As done in the literature, we restrict our attention to the class of *linear* price functions given by

$$p = \sum_{k=1}^{n} \boldsymbol{\pi}_k' \boldsymbol{y}_k + \gamma u, \tag{3}$$

where $(\boldsymbol{\pi}_1', \dots, \boldsymbol{\pi}_n')'$ represents the weights on the aggregate signal $\boldsymbol{y}$ (each component is m-dimensional, i.e., $\boldsymbol{\pi}_k = (\pi_{k1}, \dots, \pi_{km})' \in \mathbb{R}^m$ for each k), and $\gamma \neq 0$ is the weight on noise trade.[5] An equilibrium price p with the form (3) is said to be *regular* if $\gamma > 0$. The regularity means that an increase in noise demand implies an increase in the price.

3. A new approach to the existence and regularity

In this section, we propose a new approach to show the existence and regularity of linear equilibrium in the economy in last section.

Theorem 1. *There exists a linear rational expectations equilibrium.*

Proof. In the case where $\mathbf{Cov}(\theta, \boldsymbol{y}) = \mathbf{0}$, we see that $p = \frac{\mathrm{Var}(\theta)}{\sum_{k=1}^n \rho_k} u$ is an equilibrium price. We next assume that $\mathbf{Cov}(\theta, \boldsymbol{y}_i) \neq \mathbf{0}$ for at least one i.

Inserting traders' optimal demands (1) into the market-clearing condition (2), we get $\sum_{i=1}^n \frac{\mathbb{E}(\theta|\boldsymbol{y}_i, p) - p}{\rho_i \mathrm{Var}(\theta|\boldsymbol{y}_i, p)} + u = 0$. Moving the term on p into the right-hand side of the preceding equality, we obtain

$$\sum_{i=1}^{n} \frac{\mathbb{E}(\theta|\boldsymbol{y}_i, p)}{\rho_i \mathrm{Var}(\theta|\boldsymbol{y}_i, p)} + u = \sum_{i=1}^{n} \frac{1}{\rho_i \mathrm{Var}(\theta|\boldsymbol{y}_i, p)} p,$$

which can alternatively be written as

$$p = \left[\sum_{i=1}^{n} \frac{1}{\rho_i \mathrm{Var}(\theta|\boldsymbol{y}_i, p)}\right]^{-1} \left(\sum_{i=1}^{n} \frac{\mathbb{E}(\theta|\boldsymbol{y}_i, p)}{\rho_i \mathrm{Var}(\theta|\boldsymbol{y}_i, p)} + u\right). \tag{4}$$

Consider a maximal linearly independent subset of $\{\boldsymbol{y}_{ij}, i = 1, \dots, n, j = 1, \dots, m\}$, and denote the vector of such a subset by $\bar{\boldsymbol{y}}$ and its dimension by t. That is, $\mathbf{Var}(\bar{\boldsymbol{y}})$ is positive definite, and for every y_{ij} which is not in $\bar{\boldsymbol{y}}$, the variance–covariance matrix of $(\bar{\boldsymbol{y}}', y_{ij})'$ fails to have full rank.

Fix some $\ell \in \mathbb{N}$. We define an auxiliary price function

$$p_\ell(\boldsymbol{\pi}, \gamma) = \boldsymbol{\pi}'\bar{\boldsymbol{y}} + \gamma u + v/\ell, \tag{5}$$

where $\boldsymbol{\pi} \in \mathbb{R}^t$, $\gamma \in \mathbb{R}$, v is a normal random variable with mean zero and positive variance, and independent of other random variables in the model. Invoking Lemma 1 in the Appendix, we see that $\mathbb{E}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}, \gamma))$ is a linear function of $\boldsymbol{y}_i$ and $p_\ell(\boldsymbol{\pi}, \gamma)$, and is further a linear function of $\bar{\boldsymbol{y}}, u, v$ because $p_\ell(\boldsymbol{\pi}, \gamma)$ is a linear function of $\bar{\boldsymbol{y}}, u, v$ (see price function (5)), and each component of $\boldsymbol{y}_i$ can be linearly expressed in terms of the components of $\bar{\boldsymbol{y}}$ by the definition of $\bar{\boldsymbol{y}}$. Also note that $\mathrm{Var}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}, \gamma))$ is a constant (see Lemma 1). Therefore, we can reasonably define three functions $\boldsymbol{f}_\ell : \mathbb{R}^{t+1} \to \mathbb{R}^t$, $g_\ell : \mathbb{R}^{t+1} \to \mathbb{R}$ and $h_\ell : \mathbb{R}^{t+1} \to \mathbb{R}$ such that for each $\boldsymbol{\pi} \in \mathbb{R}^t$ and $\gamma \in \mathbb{R}$, $(\boldsymbol{f}_\ell(\boldsymbol{\pi}, \gamma)', g_\ell(\boldsymbol{\pi}, \gamma), h_\ell(\boldsymbol{\pi}, \gamma))'$ is the unique vector that satisfies the following equality almost surely:

$$\left[\sum_{i=1}^{n} \frac{1}{\rho_i \mathrm{Var}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}, \gamma))}\right]^{-1} \left(\sum_{i=1}^{n} \frac{\mathbb{E}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}, \gamma))}{\rho_i \mathrm{Var}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}, \gamma))} + u\right)$$
$$= \boldsymbol{f}_\ell(\boldsymbol{\pi}, \gamma)'\bar{\boldsymbol{y}} + g_\ell(\boldsymbol{\pi}, \gamma)u + h_\ell(\boldsymbol{\pi}, \gamma)v, \tag{6}$$

where the uniqueness follows from the independence of $\bar{\boldsymbol{y}}$, u and v.

Consider the two functions $\boldsymbol{f}_\ell$ and g_ℓ. We first show that they are uniformly bounded over $\mathbb{R}^{t+1}$. By virtue of Lemma 4, the Cauchy–Schwarz inequality $\mathbb{E}((\sum_{i=1}^m z_i)^2) \le m\sum_{i=1}^m \mathbb{E}(z_i^2)$, the relation $\mathrm{Var}(\mathbb{E}(\theta|\cdot)) \le \mathrm{Var}(\theta)$ (see Lemma 3 in the Appendix), and the independence of $\bar{\boldsymbol{y}}$, u and v, squaring both sides of (6) gives

$$\mathrm{Var}(\boldsymbol{f}_\ell(\boldsymbol{\pi}, \gamma)'\bar{\boldsymbol{y}}) + g_\ell(\boldsymbol{\pi}, \gamma)^2 \mathrm{Var}(u) + h_\ell(\boldsymbol{\pi}, \gamma)^2 \mathrm{Var}(v)$$
$$\le 2\,\mathrm{Var}(\theta)^2 \left[\sum_{i=1}^{n} \frac{1}{\rho_i}\right]^{-2} \left(n \sum_{i=1}^{n} \frac{\mathrm{Var}(\theta)}{\rho_i^2 \mathrm{Var}(\theta|\boldsymbol{y})^2} + \mathrm{Var}(u)\right). \tag{7}$$

Let λ_{min} denote the minimum eigenvalue of $\mathbf{Var}(\bar{\boldsymbol{y}})$, which is positive because $\mathbf{Var}(\bar{\boldsymbol{y}})$ is positive definite by the definition of $\bar{\boldsymbol{y}}$. By the relation

$$\lambda_{min}|\boldsymbol{f}_\ell(\boldsymbol{\pi}, \gamma)|^2 \le \mathrm{Var}(\boldsymbol{f}_\ell(\boldsymbol{\pi}, \gamma)'\bar{\boldsymbol{y}})$$

and the inequality (7), we see that

$$|\boldsymbol{f}_\ell(\boldsymbol{\pi}, \gamma)| \le \sqrt{\frac{2\,\mathrm{Var}(\theta)^2}{\lambda_{min}} \left[\sum_{i=1}^{n} \frac{1}{\rho_i}\right]^{-2} \left(n \sum_{i=1}^{n} \frac{\mathrm{Var}(\theta)}{\rho_i^2 \mathrm{Var}(\theta|\boldsymbol{y})^2} + \mathrm{Var}(u)\right)} =: B_1$$

and

$$|g_\ell(\boldsymbol{\pi}, \gamma)| \le \sqrt{\frac{2\,\mathrm{Var}(\theta)^2}{\mathrm{Var}(u)} \left[\sum_{i=1}^{n} \frac{1}{\rho_i}\right]^{-2} \left(n \sum_{i=1}^{n} \frac{\mathrm{Var}(\theta)}{\rho_i^2 \mathrm{Var}(\theta|\boldsymbol{y})^2} + \mathrm{Var}(u)\right)} =: B_2,$$

implying that the two functions $\boldsymbol{f}_\ell$ and g_ℓ are uniformly bounded over $\mathbb{R}^{t+1}$. Indeed, $|\boldsymbol{f}_\ell(\boldsymbol{\pi}, \gamma)| \le B$ and $|g_\ell(\boldsymbol{\pi}, \gamma)| \le B$ for any $\boldsymbol{\pi} \in \mathbb{R}^t$ and $\gamma \in \mathbb{R}$, where $B = \max\{B_1, B_2\}$. Furthermore, by Lemma 2 in the Appendix, $\boldsymbol{f}_\ell$ and g_ℓ are continuous on $\mathbb{R}^{t+1}$. It then follows from Brouwer's fixed-point theorem that the restriction of functions $\boldsymbol{f}_\ell$ and g_ℓ to the subdomain $[-B, B]^{t+1}$ has a fixed point $(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)$ (this fixed point clearly depends on ℓ, so we use the subscript ℓ to highlight it), i.e.,

$$\boldsymbol{f}_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*) = \boldsymbol{\pi}_\ell^* \text{ and } g_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*) = \gamma_\ell^*.$$

It is easy to see that $\gamma_\ell^* \neq 0$ because otherwise, on the one hand, the coefficient on u on the right-hand side of (6) equals zero, but on the other hand, the coefficient on u on the left-hand side of (6) is not less than the positive number $\left[\sum_{i=1}^n \frac{1}{\rho_i}\right]^{-1} \mathrm{Var}(\theta|\boldsymbol{y})$ (invoking Lemma 4 in the Appendix), a contradiction.

For each $\ell \in \mathbb{N}$, we can get a fixed point $(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)$ of the two functions $\boldsymbol{f}_\ell$ and g_ℓ. Consider the sequence of fixed points $\{(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)\}_{\ell \in \mathbb{N}}$. Plugging $\boldsymbol{\pi} = \boldsymbol{\pi}_\ell^*$ and $\gamma = \gamma_\ell^*$ into (6) gives

$$\left[\sum_{i=1}^{n} \frac{1}{\rho_i \mathrm{Var}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*))}\right]^{-1} \times \left(\sum_{i=1}^{n} \frac{\mathbb{E}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*))}{\rho_i \mathrm{Var}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*))} + u\right)$$

[5] Because all random variables are normalized to have mean zero, there is no intercept term in price function (3).

$$= (\boldsymbol{\pi}_\ell^*)'\bar{\boldsymbol{y}} + \gamma_\ell^* u + h_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)v. \quad (8)$$

Observe that the sequence $\{(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)\}_{\ell\in\mathbb{N}}$ is bounded noting the uniform boundedness of $\boldsymbol{f}_\ell$ and g_ℓ over $\mathbb{R}^{t+1}$, as shown earlier. Hence, we may pass to a subsequence if necessary and assume that $\boldsymbol{\pi}_\ell^* \to \boldsymbol{\pi}^*$ and $\gamma_\ell^* \to \gamma^*$ for some $\boldsymbol{\pi}^* \in \mathbb{R}^t$ and $\gamma^* \in \mathbb{R}$ as $\ell \to \infty$.

We claim now that $\boldsymbol{\pi}^* \neq \boldsymbol{0}$. Indeed, taking the covariance $Cov(\theta, \cdot)$ of both sides of (8) gives

$$\begin{aligned}
&(\boldsymbol{\pi}_\ell^*)'\mathbf{Cov}(\theta, \bar{\boldsymbol{y}}) \\
&= \left[\sum_{i=1}^n \frac{1}{\rho_i \operatorname{Var}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*))}\right]^{-1} \\
&\quad \times \sum_{i=1}^n \frac{Cov(\theta, \mathbb{E}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)))}{\rho_i \operatorname{Var}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*))} \\
&\geq \left[\sum_{i=1}^n \frac{1}{\rho_i \operatorname{Var}(\theta|\boldsymbol{y})}\right]^{-1} \frac{\min_{\{i|\mathbf{Cov}(\theta,\boldsymbol{y}_i)\neq\boldsymbol{0}\}} \epsilon_i}{\operatorname{Var}(\theta)\max_{1\leq k\leq n}\rho_k} > 0 \quad (9)
\end{aligned}$$

for every $\ell \in \mathbb{N}$, where the equality uses the independence of θ and $\{u, v\}$, the two inequalities follow from Lemmas 4 and 5 in the Appendix. Consequently, it follows from (9) that, as the limit of $\{\boldsymbol{\pi}_\ell^*\}$, $\boldsymbol{\pi}^* \neq \boldsymbol{0}$, establishing the claim.

We also claim that $\gamma^* \neq 0$. By Lemma 1 in the Appendix, the conditional mean of θ is given by

$$\mathbb{E}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)) = \boldsymbol{\alpha}_{i,\ell}'\boldsymbol{y}_i + \beta_{i,\ell}p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*), \quad (10)$$

where

$$\begin{aligned}
\boldsymbol{\alpha}_{i,\ell} &= \left[\mathbf{Var}(\boldsymbol{y}_i) - \frac{\mathbf{Cov}(\boldsymbol{\pi}_\ell^*, \boldsymbol{y}_i)\mathbf{Cov}(\boldsymbol{\pi}_\ell^*, \boldsymbol{y}_i)'}{\operatorname{Var}(\boldsymbol{\pi}_\ell^*) + (\gamma_\ell^*)^2\operatorname{Var}(u) + \operatorname{Var}(v)/\ell^2}\right]^{-1} \\
&\quad \times \left[\mathbf{Cov}(\theta, \boldsymbol{y}_i) - \frac{Cov(\boldsymbol{\pi}_\ell^*, \theta)}{\operatorname{Var}(\boldsymbol{\pi}_\ell^*) + (\gamma_\ell^*)^2\operatorname{Var}(u) + \operatorname{Var}(v)/\ell^2}\mathbf{Cov}(\boldsymbol{\pi}_\ell^*, \boldsymbol{y}_i)\right], \\
\beta_{i,\ell} &= \frac{Cov(\boldsymbol{\pi}_\ell^*, \theta) - \mathbf{Cov}(\theta, \boldsymbol{y}_i)'\mathbf{Var}(\boldsymbol{y}_i)^{-1}\mathbf{Cov}(\boldsymbol{\pi}_\ell^*, \boldsymbol{y}_i)}{\operatorname{Var}(\boldsymbol{\pi}_\ell^*) + (\gamma_\ell^*)^2\operatorname{Var}(u) + \operatorname{Var}(v)/\ell^2 - \mathbf{Cov}(\boldsymbol{\pi}_\ell^*, \boldsymbol{y}_i)'\mathbf{Var}(\boldsymbol{y}_i)^{-1}\mathbf{Cov}(\boldsymbol{\pi}_\ell^*, \boldsymbol{y}_i)}.
\end{aligned}$$

Plugging (10) into (8) and noting the relation $p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*) = (\boldsymbol{\pi}_\ell^*)'\bar{\boldsymbol{y}} + \gamma_\ell^* u + v/\ell$, we have

$$\begin{aligned}
&\left[\sum_{i=1}^n \frac{1}{\rho_i \operatorname{Var}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*))}\right]^{-1} \\
&\quad \times \left(\sum_{i=1}^n \frac{\boldsymbol{\alpha}_{i,\ell}'\boldsymbol{y}_i + \beta_{i,\ell}p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)}{\rho_i \operatorname{Var}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*))} + u\right) \\
&= p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*) - v/\ell + h_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)v,
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*) &= \left[\sum_{i=1}^n \frac{1-\beta_{i,\ell}}{\rho_i \operatorname{Var}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*))}\right]^{-1} \\
&\quad \times \left(\sum_{i=1}^n \frac{\boldsymbol{\alpha}_{i,\ell}'\boldsymbol{y}_i}{\rho_i \operatorname{Var}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*))}\right. \\
&\quad + u + \left[\sum_{i=1}^n \frac{1}{\rho_i \operatorname{Var}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*))}\right] \\
&\quad \left. \times (1/\ell - h_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*))v\right). \quad (11)
\end{aligned}$$

Define $\boldsymbol{Q}_\ell^* = \boldsymbol{\pi}_\ell^*/\gamma_\ell^*$ (notice that we have shown that $\gamma_\ell^* \neq 0$). Here without loss of generality, we assume that $\mathbf{Var}(\boldsymbol{y})$ is positive definite. Matching the coefficients on both sides of (11), we have for every $i = 1, \ldots, n$, Eq. (12) which is given in Box I. Following the arguments in the proof of Proposition 1 in Lou et al. (2019), we can show that the sequence $\{\boldsymbol{Q}_\ell^*\}$ is bounded. As a result, $\gamma^* \neq 0$ because otherwise, $\boldsymbol{\pi}^* = \boldsymbol{0}$, contradicting $\boldsymbol{\pi}^* \neq \boldsymbol{0}$ that we have shown above.

We finally show that $p = (\boldsymbol{\pi}^*)'\bar{\boldsymbol{y}} + \gamma^* u$ is an equilibrium price. Because $\gamma^* \neq 0$, the variance–covariance matrix of $(\boldsymbol{y}_i', (\boldsymbol{\pi}^*)'\bar{\boldsymbol{y}} + \gamma^* u)'$ is positive definite. Then it follows from Lemma 2 in the Appendix that

$$\mathbb{E}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)) \to \mathbb{E}(\theta|\boldsymbol{y}_i, (\boldsymbol{\pi}^*)'\bar{\boldsymbol{y}} + \gamma^* u)$$

as $\ell \to \infty$. By the preceding relation and Lemma 3 in the Appendix,

$$\operatorname{Var}(\theta|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)) \to \operatorname{Var}(\theta|\boldsymbol{y}_i, (\boldsymbol{\pi}^*)'\bar{\boldsymbol{y}} + \gamma^* u)$$

as $\ell \to \infty$. Consequently, the term on v at the left-hand side of (8) disappears and thus $h_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*) \to 0$ as $\ell \to \infty$. Taking the limit of both sides of (8) as $\ell \to \infty$, we conclude that $p = (\boldsymbol{\pi}^*)'\bar{\boldsymbol{y}} + \gamma^* u$ satisfies (4) and is thus a linear equilibrium price. The proof is completed. □

We summarize the new approach to the proof of the equilibrium existence in Theorem 1 as follows. First, we construct an auxiliary price function in which an independent random variable is additionally introduced; see (5), and write the market-clearing condition in an alternative form where one side is the price function while the other side is a term involving a variance-adjusted conditional expectation; see (4). Second, we construct two functions (i.e., $\boldsymbol{f}$ and g) which map to the coefficients on signals and noise trade in the price function based on an alternative form of the market-clearing condition, and show the uniform boundedness and continuity of the two functions based on some elegant properties of conditional expectation and variance. Third, applying Brouwer's fixed-point theorem to get a sequence of fixed points (i.e., $\{(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)\}$). Fourth, in order to take the limit as the introduced random variable vanishes (i.e., $\ell \to \infty$) to get the desired equilibrium, we show that any limit point of the coefficients on the signals and noise trade do not equal zero. Finally, taking the limit of both sides of the alternative market-clearing condition gives a linear equilibrium.

The main reason why we introduce an auxiliary random variable into the price function is that the coefficient on noise trade u in the expression $\mathbb{E}(\theta|\boldsymbol{y}_i, \boldsymbol{\pi}'\bar{\boldsymbol{y}} + \gamma u)$ is not continuous at $\gamma = 0$ in general. So even though we get the uniform boundedness of the two functions $\boldsymbol{f}$ and g, we also cannot apply Brouwer's fixed-point theorem to get a linear equilibrium. However, the independence of the newly introduced random variable in the price function guarantees the continuity of functions $\boldsymbol{f}$ and g, and then enables us to apply Brouwer's fixed-point theorem.

The newly developed method differs from the one in Lou et al. (2019). The method in Lou et al. (2019) transfers equivalently the equilibrium existence problem to a fixed-point existence problem of a system of nonlinear algebraic equations, while our new approach is operated directly on an alternative form of market-clearing conditions. Although, similar to Lou et al. (2019), we also construct a function and need to find a fixed point of the function, we analyze the properties of the constructed function (necessary for applying Brouwer's fixed-point theorem), for example, uniform boundedness and continuity, directly based on the *one-dimensional*, alternative form of market-clearing conditions. Working directly on the one-dimensional market-clearing condition facilitates us to utilize some elegant properties of conditional expectation and variance, for example, the boundedness and monotonicity of conditional variance with respect to traders' signals, the law of total variance, etc. One advantage of the new method is that besides homogeneous-valuation economies, it can also handle the existence of equilibrium in economies with heterogeneous valuations, see Section 4 for details.

$$\mathbf{Q}^*_{i,\ell} = \frac{\boldsymbol{\alpha}_{i,\ell}}{\rho_i \operatorname{Var}(\theta|\mathbf{y}_i, p_\ell(\mathbf{Q}^*_\ell, \gamma^*_\ell))}$$
$$= \frac{\left[\mathbf{Var}(\mathbf{y}_i) - \frac{\mathbf{Cov}(\mathbf{Q}^*_\ell, \mathbf{y}_i)\mathbf{Cov}(\mathbf{Q}^*_\ell, \mathbf{y}_i)'}{\operatorname{Var}(\mathbf{Q}^*_\ell)+\operatorname{Var}(u)+\operatorname{Var}(v)/(\ell\gamma^*_\ell)^2}\right]^{-1}\left[\mathbf{Cov}(\theta, \mathbf{y}_i) - \frac{Cov(\mathbf{Q}^*_\ell, \theta)}{\operatorname{Var}(\mathbf{Q}^*_\ell)+\operatorname{Var}(u)+\operatorname{Var}(v)/(\ell\gamma^*_\ell)^2}\mathbf{Cov}(\mathbf{Q}^*_\ell, \mathbf{y}_i)\right]}{\rho_i \operatorname{Var}(\theta|\mathbf{y}_i, (\mathbf{Q}^*_\ell)'\bar{\mathbf{y}} + u + v/(\ell\gamma^*_\ell))}. \tag{12}$$

Box I.

Theorem 1 shows the existence, but not the regularity of linear equilibrium.[6] The following theorem establishes the regularity of linear equilibrium when noise trading is relatively large or signals take the special form of the sum of the fundamental and an independent noise.

Theorem 2. *Suppose either of the following two conditions holds:*

- $\operatorname{Var}(u) > n\sum_{i=1}^n \frac{\operatorname{Var}(\theta)}{\rho_i^2 \operatorname{Var}(\theta|\mathbf{y})^2}$;
- *Traders' signals take the form of $y_{ij} = \theta + \epsilon_{ij}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, where any two signals are either identical or have mutually independent noises.*

Then there exists a regular, linear rational expectations equilibrium.

Proof. In this proof, we follow the notations used in the proof of Theorem 1. We first consider the first condition. We claim that $\inf_{\ell\in\mathbb{N}} \gamma^*_\ell > 0$. Because $\mathbb{E}(\theta|\mathbf{y}_i, p_\ell(\boldsymbol{\pi}^*_\ell, \gamma^*_\ell))$ is a linear function of $\mathbf{y}_i$ and $p_\ell(\boldsymbol{\pi}^*_\ell, \gamma^*_\ell)$ by Lemma 1 in the Appendix, and $p_\ell(\boldsymbol{\pi}^*_\ell, \gamma^*_\ell)$ is further a linear function of $\bar{\mathbf{y}}$, u and v, there is a term on u and denote such a coefficient as $v_\ell(\boldsymbol{\pi}^*_\ell, \gamma^*_\ell)$ in the expression $\sum_{i=1}^n \frac{\mathbb{E}(\theta|\mathbf{y}_i, p_\ell(\boldsymbol{\pi}^*_\ell, \gamma^*_\ell))}{\rho_i \operatorname{Var}(\theta|\mathbf{y}_i, p_\ell(\boldsymbol{\pi}^*_\ell, \gamma^*_\ell))}$. We have

$$\begin{aligned} v_\ell(\boldsymbol{\pi}^*_\ell, \gamma^*_\ell)^2 \operatorname{Var}(u) &\le \operatorname{Var}\left(\sum_{i=1}^n \frac{\mathbb{E}(\theta|\mathbf{y}_i, p_\ell(\boldsymbol{\pi}^*_\ell, \gamma^*_\ell))}{\rho_i \operatorname{Var}(\theta|\mathbf{y}_i, p_\ell(\boldsymbol{\pi}^*_\ell, \gamma^*_\ell))}\right) \\ &\le n\sum_{i=1}^n \frac{\operatorname{Var}(\mathbb{E}(\theta|\mathbf{y}_i, p_\ell(\boldsymbol{\pi}^*_\ell, \gamma^*_\ell)))}{\rho_i^2 \operatorname{Var}(\theta|\mathbf{y}_i, p_\ell(\boldsymbol{\pi}^*_\ell, \gamma^*_\ell))^2} \\ &\le n\sum_{i=1}^n \frac{\operatorname{Var}(\theta)}{\rho_i^2 \operatorname{Var}(\theta|\mathbf{y})^2}, \end{aligned} \tag{13}$$

where the first inequality follows from the independence of u, v and $\mathbf{y}$, the second from the Cauchy–Schwarz inequality $\mathbb{E}((\sum_{i=1}^m z_i)^2) \le m\sum_{i=1}^m \mathbb{E}(z_i^2)$, and the last one from Lemmas 3 and 4 in the Appendix. Then by (13), together with the condition on $\operatorname{Var}(u)$ in this theorem, we have

$$|v_\ell(\boldsymbol{\pi}^*_\ell, \gamma^*_\ell)| \le \sqrt{\frac{n\sum_{i=1}^n \frac{\operatorname{Var}(\theta)}{\rho_i^2 \operatorname{Var}(\theta|\mathbf{y})^2}}{\operatorname{Var}(u)}} < 1 \tag{14}$$

for any $\ell \in \mathbb{N}$. Together, (8) and (14) imply that

$$\begin{aligned} \gamma^*_\ell &\ge \left[\sum_{i=1}^n \frac{1}{\rho_i \operatorname{Var}(\theta|\mathbf{y}_i, p_\ell(\boldsymbol{\pi}^*_\ell, \gamma^*_\ell))}\right]^{-1} \left(1 - |\, v_\ell\left(\boldsymbol{\pi}^*_\ell, \gamma^*_\ell\right|\right) \\ &\ge \left[\sum_{i=1}^n \frac{1}{\rho_i \operatorname{Var}(\theta|\mathbf{y})}\right]^{-1}\left(1 - \sqrt{\frac{n\sum_{i=1}^n \frac{\operatorname{Var}(\theta)}{\rho_i^2 \operatorname{Var}(\theta|\mathbf{y})^2}}{\operatorname{Var}(u)}}\right) \\ &> 0 \end{aligned}$$

for every $\ell \in \mathbb{N}$. This provides a strictly positive, uniform lower bound to the sequence $\{\gamma^*_\ell\}$. As a result, as the limit of $\{\gamma^*_\ell\}$, we have $\gamma^* > 0$. Using the similar arguments as in the proof of Theorem 1, we can show that there exists a linear equilibrium. The linear equilibrium is clearly regular because $\gamma^* > 0$.

Next, we consider the special signal structure of $y_{ij} = \theta + \epsilon_{ij}$. Denote $\tau_{ij} = 1/\operatorname{Var}(\epsilon_{ij})$. Take $\ell = \infty$ such that there is no term v in the price function, and denote $\boldsymbol{f}_\infty$ and g_∞ as $\boldsymbol{f}$ and g for simplicity, respectively. Consider the expression $\mathbb{E}(\theta|\mathbf{y}_i, \boldsymbol{\pi}'\bar{\mathbf{y}}+\gamma u)$, where $\boldsymbol{\pi} \in \mathbb{R}^t_+$ and $\gamma \in \mathbb{R}_+$. Without loss of generality, we assume that $\bar{\mathbf{y}}$ does not include the random variables in $\mathbf{y}_i$ (otherwise, we can remove the components of $\boldsymbol{\pi}'\bar{\mathbf{y}}$ from $\boldsymbol{\pi}'\bar{\mathbf{y}} + \gamma u$ which can be linearly expressed in terms of the components of $\mathbf{y}_i$). For notational simplicity, we re-denote the random variables in $\bar{\mathbf{y}}$ as $\{y_k = \theta + \epsilon_k, k = 1, \ldots, t\}$. Because $\mathbf{Var}(\bar{\mathbf{y}})$ has a full rank and it is assumed that any two signals are either identical or have mutually independent noises, the noise terms $\{\epsilon_k\}$ are mutually independent. By Lemma 1 in the Appendix, together with some simple calculations, we get

$$\mathbb{E}(\theta|\mathbf{y}_i, \boldsymbol{\pi}'\bar{\mathbf{y}}+\gamma u) = \frac{\sum_{j=1}^m \tau_{ij}y_{ij} + \frac{\sum_{k=1}^t \pi_k}{\sum_{k=1}^t \pi_k^2 \operatorname{Var}(\epsilon_k)+\gamma^2 \operatorname{Var}(u)}(\boldsymbol{\pi}'\bar{\mathbf{y}} + \gamma u)}{\frac{1}{\operatorname{Var}(\theta)} + \sum_{j=1}^m \tau_{ij} + \frac{(\sum_{k=1}^t \pi_k)^2}{\sum_{k=1}^t \pi_k^2 \operatorname{Var}(\epsilon_k)+\gamma^2 \operatorname{Var}(u)}}$$

and

$$\operatorname{Var}(\theta|\mathbf{y}_i, \boldsymbol{\pi}'\bar{\mathbf{y}} + \gamma u) = \frac{1}{\frac{1}{\operatorname{Var}(\theta)} + \sum_{j=1}^m \tau_{ij} + \frac{(\sum_{k=1}^t \pi_k)^2}{\sum_{k=1}^t \pi_k^2 \operatorname{Var}(\epsilon_k)+\gamma^2 \operatorname{Var}(u)}}.$$

Denote $\tau_{min} = \min_{1\le i\le n, 1\le j\le m} \tau_{ij} > 0$. Then we see that for any $\boldsymbol{\pi} \in \mathbb{R}^t_+$, $\gamma \in \mathbb{R}_+$ and $k = 1, \ldots, t$,

$$\boldsymbol{f}(\boldsymbol{\pi}, \gamma)_k \ge \operatorname{Var}(\theta|\mathbf{y})\left[\sum_{i=1}^n \frac{1}{\rho_i}\right]^{-1} \frac{\tau_{min}}{\max_{1\le i\le n} \rho_i} =: \eta_1 > 0$$

and

$$g(\boldsymbol{\pi}, \gamma) \ge \operatorname{Var}(\theta|\mathbf{y})\left[\sum_{i=1}^n \frac{1}{\rho_i}\right]^{-1} =: \eta_2 > 0,$$

where $\boldsymbol{f}(\boldsymbol{\pi}, \gamma)_k$ denotes the kth component of vector $\boldsymbol{f}(\boldsymbol{\pi}, \gamma)$. By Lemma 1 in the Appendix, $\boldsymbol{f}$ and g are continuous at every $(\boldsymbol{\pi}, \gamma)$ with $\gamma > 0$.[7] Using Brouwer's fixed-point theorem with the restriction of functions $\boldsymbol{f}$ and g to the subdomain $[\eta_1, B]^t \times [\eta_2, B]$, we get a regular, linear rational expectations equilibrium. □

Remark 1. Theorem 1 establishes the nonzero of the limit points of $\{\gamma^*_\ell\}$ indirectly by first showing the boundedness of the sequence $\{\mathbf{Q}^*_\ell\}$, then using the relation $\boldsymbol{\pi}^*_\ell = \gamma^*_\ell \mathbf{Q}^*_\ell$ and the nonzero of the limit points of $\{\boldsymbol{\pi}^*_\ell\}$. Differently, under some conditions

[6] Indeed, in the proof of Theorem 1, we only show that $\gamma^* \neq 0$, but not $\gamma^* > 0$.

[7] The continuity is guaranteed by the special structure of signals being a sum of the fundamental and an independent noise. So unlike the proof of Theorem 1, here an auxiliary normal random variable v is no longer needed to be introduced in price functions.

Theorem 2 directly shows that every γ_ℓ^* is strictly larger than some positive number. Consequently, any limit point of $\{\gamma_\ell^*\}$ is strictly positive and the resulting equilibrium is thus regular. The method used here also differs from the one in Lou ⓡ al. (2019) which establishes the regularity of linear equilibrium based on the explicit expression $\gamma = \gamma(\boldsymbol{Q})$, which is a nonlinear function of some fixed point $\boldsymbol{Q}$, together with some properties of the fixed-point equation (similar to (12)) that the fixed point $\boldsymbol{Q}$ satisfies (refer to the proof of Proposition 2 in Lou ⓡ al. (2019) for more details).

Remark 2. For any exogenously given signals $y_i = \theta + \epsilon_i$ (where the noise terms ϵ_i's are mutually independent), when traders share signals with their neighbors in a social network, the resulting new signals of traders satisfy the second condition in Theorem 2.

As a byproduct of the newly developed approach, we present an information aggregation result at the end of this section, which recovers Proposition 5 in Lou ⓡ al. (2019). The information aggregation result reveals that the equilibrium price is able to aggregate in a good manner the diverse information in the economy.

Theorem 3. *For any linear equilibrium price p, it holds that $Cov(\theta, p) \geq 0$, and the strict inequality holds if and only if* $\mathbf{Cov}(\theta, \boldsymbol{y}_i) \neq \mathbf{0}$ *for some i.*

Proof. Recall the alternative form of the market-clearing condition

$$p = \left[\sum_{i=1}^n \frac{1}{\rho_i \operatorname{Var}(\theta|\boldsymbol{y}_i, p)}\right]^{-1} \left(\sum_{i=1}^n \frac{\mathbb{E}(\theta|\boldsymbol{y}_i, p)}{\rho_i \operatorname{Var}(\theta|\boldsymbol{y}_i, p)} + u\right). \tag{15}$$

Taking the covariance $Cov(\theta, \cdot)$ of both sides of (15), we have

$$\begin{aligned} Cov(\theta, p) &= \left[\sum_{i=1}^n \frac{1}{\rho_i \operatorname{Var}(\theta|\boldsymbol{y}_i, p)}\right]^{-1} \sum_{i=1}^n \frac{Cov(\theta, \mathbb{E}(\theta|\boldsymbol{y}_i, p))}{\rho_i \operatorname{Var}(\theta|\boldsymbol{y}_i, p)} \\ &= \operatorname{Var}(\theta)\left[\sum_{i=1}^n \frac{1}{\rho_i \operatorname{Var}(\theta|\boldsymbol{y}_i, p)}\right]^{-1} \sum_{i=1}^n \frac{1 - \frac{\operatorname{Var}(\theta|\boldsymbol{y}_i, p)}{\operatorname{Var}(\theta)}}{\rho_i \operatorname{Var}(\theta|\boldsymbol{y}_i, p)} \\ &= \operatorname{Var}(\theta)\left(1 - \left[\sum_{i=1}^n \frac{1}{\rho_i \operatorname{Var}(\theta|\boldsymbol{y}_i, p)}\right]^{-1} \sum_{i=1}^n \frac{1}{\rho_i \operatorname{Var}(\theta)}\right) \\ &\geq 0, \end{aligned} \tag{16}$$

where the first equality uses the independence of θ and u, the second uses the relation $Cov(\theta, \mathbb{E}(\theta|\boldsymbol{y}_i, p)) = \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i, p))$ (Lemma 5 in the Appendix) and Lemma 3, and the inequality follows from the relation $\operatorname{Var}(\theta|\boldsymbol{y}_i, p) \leq \operatorname{Var}(\theta)$ (see Lemma 3 again). The weak inequality follows.

Observing (16), we immediately conclude that $Cov(\theta, p) = 0$ if and only if $\operatorname{Var}(\theta|\boldsymbol{y}_i, p) = \operatorname{Var}(\theta)$ for every i. It follows from Lemma 1 that $\operatorname{Var}(\theta|\boldsymbol{y}_i, p) = \operatorname{Var}(\theta)$ if and only if $\mathbf{Cov}(\theta, \boldsymbol{y}_i) = \mathbf{0}$ for every i, so the conclusion about strict inequality follows. □

Remark 3. The equality (16) presents an intrinsic relationship between the covariance $Cov(\theta, p)$ and the model parameters such as the risk aversion coefficients, the prior variance and the posterior conditional variance of the fundamental. It can be seen from (16) that the covariance $Cov(\theta, p)$ is upper bounded by the prior variance $\operatorname{Var}(\theta)$. An interesting observation from (16) is that when some trader's signal $\boldsymbol{y}_i$ is sufficiently informative in the sense that $\operatorname{Var}(\theta|\boldsymbol{y}_i)$ is close to zero, the covariance $Cov(\theta, p)$ will be close to its upper bound $\operatorname{Var}(\theta)$. This means that the equilibrium price is able to aggregate in a good manner the diverse information in the economy.

4. An extended model with heterogeneous valuations

We now consider an extension of the model in Section 2 that traders valuate the risky asset *heterogeneously*. Each trader i's valuation of the risky asset is described by a random variable θ_i, which is normally distributed with mean zero and positive variance. Except for the heterogeneity of traders' valuations, all the settings in the extended model are the same as that in the economy in Section 2. Under the heterogeneous-valuation setting, the optimal demand for the risky asset by trader i is given by

$$x_i^* = \frac{\mathbb{E}(\theta_i|\boldsymbol{y}_i, p) - p}{\rho_i \operatorname{Var}(\theta_i|\boldsymbol{y}_i, p)},$$

and the market-clearing condition becomes

$$\sum_{i=1}^n \frac{\mathbb{E}(\theta_i|\boldsymbol{y}_i, p) - p}{\rho_i \operatorname{Var}(\theta_i|\boldsymbol{y}_i, p)} + u = 0.$$

Similar to Assumption 1 for the homogeneous-valuation economy in Section 2, here we also impose a mild assumption that the aggregate signal cannot fully pin down any trader's valuation.

Assumption 2. For every i, $\operatorname{Var}(\theta_i|\boldsymbol{y}) > 0$.

When traders valuate the risky asset homogeneously, i.e., $\theta_i = \theta_j$ for all i and j, the economy considered in this section coincides with the one in Section 2. We next show that the proposed approach in Theorems 1 and 2 can also be applied here to solve the existence and regularity of linear equilibrium in heterogeneous-valuation economies.

Theorem 4. *Consider the heterogeneous-valuation economy in this section with Assumption 2. Suppose* $\operatorname{Var}(u) > n\sum_{i=1}^n \frac{\operatorname{Var}(\theta_i)}{\rho_i^2 \operatorname{Var}(\theta_i|\boldsymbol{y})^2}$. *Then there exists a regular, linear rational expectations equilibrium.*

Proof. When $\sum_{i=1}^n \frac{\mathbb{E}(\theta_i|\boldsymbol{y}_i)}{\rho_i \operatorname{Var}(\theta_i|\boldsymbol{y}_i)} = 0$, we can see that $p = \left[\frac{1}{\sum_{i=1}^n \rho_i \operatorname{Var}(\theta_i|\boldsymbol{y}_i)}\right]^{-1} u$ is an equilibrium price. Note that

$$\operatorname{Var}(\theta_i|\boldsymbol{y}_i) \geq \operatorname{Var}(\theta_i|\boldsymbol{y}) > 0$$

for every i by Assumption 2. We next assume that $\sum_{i=1}^n \frac{\mathbb{E}(\theta_i|\boldsymbol{y}_i)}{\rho_i \operatorname{Var}(\theta_i|\boldsymbol{y}_i)} \neq 0$, and borrow the arguments and follow the notations used in the proof of Theorems 1 and 2.

Define three functions $\boldsymbol{f}_\ell : \mathbb{R}^{t+1} \to \mathbb{R}^t$, $g_\ell : \mathbb{R}^{t+1} \to \mathbb{R}$ and $h_\ell : \mathbb{R}^{t+1} \to \mathbb{R}$ such that for any $\boldsymbol{\pi} \in \mathbb{R}^t$ and $\gamma \in \mathbb{R}$, $(\boldsymbol{f}_\ell(\boldsymbol{\pi}, \gamma)', g_\ell(\boldsymbol{\pi}, \gamma), h_\ell(\boldsymbol{\pi}, \gamma))'$ is the unique vector that satisfies the following equality almost surely

$$\begin{aligned} &\left[\sum_{i=1}^n \frac{1}{\rho_i \operatorname{Var}(\theta_i|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}, \gamma))}\right]^{-1} \left(\sum_{i=1}^n \frac{\mathbb{E}(\theta_i|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}, \gamma))}{\rho_i \operatorname{Var}(\theta_i|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}, \gamma))} + u\right) \\ &= \boldsymbol{f}_\ell(\boldsymbol{\pi}, \gamma)'\bar{\boldsymbol{y}} + g_\ell(\boldsymbol{\pi}, \gamma)u + h_\ell(\boldsymbol{\pi}, \gamma)v. \end{aligned} \tag{17}$$

Similar to the arguments used in the proof of Theorem 1, we can show that for every $\ell \in \mathbb{N}$, $\boldsymbol{f}_\ell$ and g_ℓ are uniformly bounded over $\mathbb{R}^{t+1}$, continuous at every point in $\mathbb{R}^{t+1}$ and then have a fixed point $(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)$ such that $\boldsymbol{f}_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*) = \boldsymbol{\pi}_\ell^*$ and $g_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*) = \gamma_\ell^*$ by using Brouwer's fixed-point theorem. Let $(\boldsymbol{\pi}^*, \gamma^*)$ be the limit of the sequence $\{(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)\}$ (otherwise take a subsequence of $\mathbb{N}$ due to the boundedness of $\{(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)\}$ by the uniform boundedness of $\boldsymbol{f}_\ell$ and g_ℓ).

Let $\upsilon_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)$ denote the coefficient on u in the expression $\sum_{i=1}^n \frac{\mathbb{E}(\theta_i|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*))}{\rho_i \operatorname{Var}(\theta_i|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*))}$; refer to the explanations on the existence of the term on u in Theorem 2. Applying similar arguments given in the proof of Theorem 2, we get

$$\upsilon_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)^2 \operatorname{Var}(u) \leq \operatorname{Var}\left(\sum_{i=1}^n \frac{\mathbb{E}(\theta_i|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*))}{\rho_i \operatorname{Var}(\theta_i|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*))}\right) \leq n \sum_{i=1}^n \frac{\operatorname{Var}(\theta_i)}{\rho_i^2 \operatorname{Var}(\theta_i|\boldsymbol{y})^2},$$

implying that

$$|\upsilon_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)| \leq \sqrt{\frac{n\sum_{i=1}^n \frac{\operatorname{Var}(\theta_i)}{\rho_i^2 \operatorname{Var}(\theta_i|\boldsymbol{y})^2}}{\operatorname{Var}(u)}} < 1$$

for any $\ell \in \mathbb{N}$, where the strict inequality follows from the condition on $\operatorname{Var}(u)$ given in this theorem. Hence, by the preceding inequality and (17), we have for every $\ell \in \mathbb{N}$,

$$\gamma_\ell^* \geq \left[\sum_{i=1}^n \frac{1}{\rho_i \operatorname{Var}(\theta_i|\boldsymbol{y})}\right]^{-1} \left(1 - \sqrt{\frac{n\sum_{i=1}^n \frac{\operatorname{Var}(\theta_i)}{\rho_i^2 \operatorname{Var}(\theta_i|\boldsymbol{y})^2}}{\operatorname{Var}(u)}}\right) > 0.$$

Hence, as the limit point of $\{\gamma_\ell^*\}$, $\gamma^* > 0$.

Furthermore, we also claim that $\boldsymbol{\pi}^* \neq \mathbf{0}$ because otherwise, with the fact that $\gamma^* > 0$ in mind and applying Lemma 2, we have $\mathbb{E}(\theta_i|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)) \to \mathbb{E}(\theta_i|\boldsymbol{y}_i, \gamma^* u) = \mathbb{E}(\theta_i|\boldsymbol{y}_i)$ almost surely. Then it follows from Lemma 3 that $\operatorname{Var}(\theta_i|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)) \to \operatorname{Var}(\theta_i|\boldsymbol{y}_i)$. Consequently, first taking the limit of both sides of (17) and then matching the coefficients on $\boldsymbol{y}$ give $\sum_{i=1}^n \frac{\mathbb{E}(\theta_i|\boldsymbol{y}_i)}{\rho_i \operatorname{Var}(\theta_i|\boldsymbol{y}_i)} = 0$, contradicting the condition $\sum_{i=1}^n \frac{\mathbb{E}(\theta_i|\boldsymbol{y}_i)}{\rho_i \operatorname{Var}(\theta_i|\boldsymbol{y}_i)} \neq 0$ assumed at the beginning of the proof. As a result, $\boldsymbol{\pi}^* \neq \mathbf{0}$.

Similar to the arguments used in the proof of Theorem 1, we have

$$\mathbb{E}(\theta_i|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)) \to \mathbb{E}(\theta_i|\boldsymbol{y}_i, (\boldsymbol{\pi}^*)'\bar{\boldsymbol{y}} + \gamma^* u),$$

$$\operatorname{Var}(\theta_i|\boldsymbol{y}_i, p_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*)) \to \operatorname{Var}(\theta_i|\boldsymbol{y}_i, (\boldsymbol{\pi}^*)'\bar{\boldsymbol{y}} + \gamma^* u)$$

and $h_\ell(\boldsymbol{\pi}_\ell^*, \gamma_\ell^*) \to 0$ as $\ell \to \infty$. Taking the limit of both sides of (17) gives a regular, linear rational expectations equilibrium. □

Remark 4. Lou ⓡ al. (2019) first show the existence and then the regularity of linear equilibrium for their homogeneous-valuation economy. Unlike it, the existence and regularity of linear equilibrium for heterogeneous-valuation economies are shown simultaneously (under the condition that the variance of noise trade is large).

Theorem 4 shows that the proposed approach in Section 2 can also be used to prove the existence and regularity of linear equilibrium in heterogeneous-valuation economies. The equilibrium existence of the homogeneous-valuation economy (in absence of idiosyncratic noise) in Lou ⓡ al. (2019) [8] is shown by first constructing an auxiliary sequence of uniformly bounded functions, then getting a fixed point for every such a function, and finally taking the limit of this sequence of fixed points (which is bounded) to get a fixed point of the original function which is constructed based on coefficient matching of market-clearing conditions. The boundedness of the sequence of fixed points depends crucially on the following property: For any vector $\boldsymbol{\mu} = (\boldsymbol{\mu}_1', \ldots, \boldsymbol{\mu}_n')' \in \mathbb{R}^{nm}$, it holds that

$$\sum_{i=1}^n \boldsymbol{\mu}_i' \left(\operatorname{Var}(\boldsymbol{\mu})\mathbf{Cov}(\theta, \boldsymbol{y}_i) - Cov(\boldsymbol{\mu}, \theta)\mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{y}_i)\right) = 0$$

with θ being the fundamental of the risky asset, where the equality follows from the two relations

$$\sum_{i=1}^n \boldsymbol{\mu}_i'\mathbf{Cov}(\theta, \boldsymbol{y}_i) = Cov(\boldsymbol{\mu}, \theta)$$

and

$$\sum_{i=1}^n \boldsymbol{\mu}_i'\mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{y}_i) = \operatorname{Var}(\boldsymbol{\mu});$$

see Lemma 2 in Lou ⓡ al. (2019). However, the above property does *not* hold for heterogeneous-valuation economies in general. To be more precise,

$$\sum_{i=1}^n \boldsymbol{\mu}_i' \left(\operatorname{Var}(\boldsymbol{\mu})\mathbf{Cov}(\theta_i, \boldsymbol{y}_i) - Cov(\boldsymbol{\mu}, \theta_i)\mathbf{Cov}(\boldsymbol{\mu}, \boldsymbol{y}_i)\right)$$

generally does not equal zero regardless of whether the condition on $\operatorname{Var}(u)$ in Theorem 4 holds. As a summary, the method used in the proof of equilibrium existence for the homogeneous-valuation economy in Lou ⓡ al. (2019) cannot be applied here to solve the equilibrium existence in the heterogeneous-valuation economy considered here.

5. Concluding remarks

A new approach was proposed to show the existence and regularity of linear equilibrium in the REE economy in Lou ⓡ al. (2019). Different from the existing method which essentially requires to find a fixed point of a system of nonlinear algebraic equations, the new approach is operated directly on an alternative form of market-clearing conditions. The new approach can also be used to show the existence and regularity of an extended heterogeneous-valuation economy where the method in Lou ⓡ al. (2019) fails to work.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

In this appendix, we introduce several useful lemmas. The first one characterizes the conditional distribution of some of the components of a multivariate normal random variable while the values of the other components are given; see Chapter 5, Section 4 in DeGroot (1970) for more details.

Lemma 1. *For a normal random vector $(z, \boldsymbol{s}')'$ with mean zero and positive definite variance–covariance matrix*

$$\begin{pmatrix} \operatorname{Var}(z) & \mathbf{Cov}(z, \boldsymbol{s})' \\ \mathbf{Cov}(z, \boldsymbol{s}) & \mathbf{Var}(\boldsymbol{s}) \end{pmatrix},$$

the conditional mean is given by

$$\mathbb{E}(z|\boldsymbol{s}) = \mathbf{Cov}(z, \boldsymbol{s})'\mathbf{Var}(\boldsymbol{s})^{-1}\boldsymbol{s}.$$

The conditional variance is a constant and given by

$$\operatorname{Var}(z|\boldsymbol{s}) = \operatorname{Var}(z) - \mathbf{Cov}(z, \boldsymbol{s})'\mathbf{Var}(\boldsymbol{s})^{-1}\mathbf{Cov}(z, \boldsymbol{s}).$$

[8] The homogeneous-valuation economy in Lou ⓡ al. (2019) (without idiosyncratic noise) is mathematically equivalent to the economy in Section 2, as explained earlier.

Lemma 2. *Let z be a normal random variable with positive variance, $\boldsymbol{x}$ and $\boldsymbol{s} \in \mathbb{R}^m$ be two normal random vectors, all with mean zero, and $\{\boldsymbol{a}_k\}$ be a vector sequence in $\mathbb{R}^m$ with $\lim_{k\to\infty} \boldsymbol{a}_k = \boldsymbol{a}_*$. Suppose the variance–covariance matrix of $(\boldsymbol{x}', \boldsymbol{a}_*'\boldsymbol{s})'$ is positive definite. Then*

$$\lim_{k\to\infty} \mathbb{E}(z|\boldsymbol{x}, \boldsymbol{a}_k'\boldsymbol{s}) = \mathbb{E}(z|\boldsymbol{x}, \boldsymbol{a}_*'\boldsymbol{s})$$

almost surely.

Proof. Follows directly from Lemma 1 and the fact that if the matrix sequence $\{A_k\}$ (where each A_k is invertible) converges to an invertible matrix A_*, then $\lim_{k\to\infty} A_k^{-1} = A_*^{-1}$. □

The first part of the next lemma comes from Theorem 7 on page 159 in Mood et al. (1974), which is referred to as *the law of total variance* in the literature, and the second part comes from the first part and Lemma 1.

Lemma 3. *For any random variable z, we have*

$$\mathrm{Var}(z) = \mathrm{Var}(\mathbb{E}(z|\cdot)) + \mathbb{E}(\mathrm{Var}(z|\cdot)).$$

In particular, when all involved random variables are normally distributed, the conditional variance $\mathrm{Var}(z|\cdot)$ is a constant, and consequently, the above formula in this lemma reduces to

$$\mathrm{Var}(z) = \mathrm{Var}(\mathbb{E}(z|\cdot)) + \mathrm{Var}(z|\cdot).$$

The next one is due to Lou ⓡ al. (2019) (see Lemma 1 therein).

Lemma 4. *For each i, when $\mathbf{Cov}(\theta, \boldsymbol{y}_i) \neq \mathbf{0}$, there exists $\epsilon_i > 0$ such that for any linear price function p, it holds that*

$$0 < \mathrm{Var}(\theta|\boldsymbol{y}) \leq \mathrm{Var}(\theta|\boldsymbol{y}_i, p) \leq \mathrm{Var}(\theta) - \epsilon_i.$$

Lemma 5. *For each i, when $\mathbf{Cov}(\theta, \boldsymbol{y}_i) \neq \mathbf{0}$, there exists $\epsilon_i > 0$ such that for any linear price function p, it holds that*

$$Cov(\theta, \mathbb{E}(\theta|\boldsymbol{y}_i, p)) = \mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i, p)) > \epsilon_i.$$

Proof. The equality follows from the following series of equalities

$$\begin{aligned} Cov(\theta, \mathbb{E}(\theta|\boldsymbol{y}_i, p)) &= \mathbb{E}(\theta\mathbb{E}(\theta|\boldsymbol{y}_i, p)) = \mathbb{E}(\mathbb{E}(\theta\mathbb{E}(\theta|\boldsymbol{y}_i, p)|\boldsymbol{y}_i, p)) \\ &= \mathbb{E}(\mathbb{E}(\theta|\boldsymbol{y}_i, p)^2) = \mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i, p)), \end{aligned}$$

where the first and last equalities follow from the definitions of covariance and variance, and the assumption that the mean of θ equals zero, the second from the definition of conditional expectation, and the third from the property "pulling out what's known" of conditional expectation. By Lemma 3 in this Appendix, we have

$$\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i, p)) = \mathrm{Var}(\theta) - \mathrm{Var}(\theta|\boldsymbol{y}_i, p). \tag{18}$$

Then the strict inequality follows from (18) and Lemma 4. □

References

Barlevy, G., Veronesi, P., 2000. Information acquisition in financial markets. Rev. Econom. Stud. 67, 79–90.

Breon-Drish, B., 2015. On existence and uniqueness of equilibrium in a class of noisy rational expectations models. Rev. Econom. Stud. 82, 868–921.

Carpio, R., Guo, M., 2019. On equilibrium existence in a finite-agent, multi-asset noisy rational expectations economy. B. E. J. Theor. Econ. 20180144.

Chabakauri, G., Yuan, K., Zachariadis, K.E., 2017. Multi-Asset Noisy Rational Expectations Equilibrium with Contingent Claims. Working paper, Available at SSRN: https://ssrn.com/abstract=2446873.

DeGroot, M., 1970. Optimal Statistical Decisions. McGraw-Hall, New York.

Grossman, S., 1976. On the efficiency of competitive stock markets where traders have diverse information. J. Finance 31, 573–585.

Grossman, S.J., Stiglitz, J.E., 1980. On the impossibility of informationally efficient markets. Amer. Econ. Rev. 70, 393–408.

Hellwig, M.F., 1980. On the aggregation of information in competitive markets. J. Econom. Theory 22, 477–498.

Lou, Y. ⓡ Parsa, S. ⓡ Ray, D. ⓡ Li, D. ⓡ Wang, S., 2019. Information aggregation in a financial market with general signal structure. J. Econom. Theory 183, 594–624.

Mood, A., Graybill, F., Boes, D., 1974. Introduction to the Theory of Statistics, third ed. McGraw-Hill.

Ozsoylev, H.N., Walden, J., 2011. Asset pricing in large information networks. J. Econom. Theory 146, 2252–2280.

Pálvölgyi, D., Venter, G., 2015a. Multiple Equilibria in Noisy Rational Expectations Economies. Working paper, Available at SSRN: https://ssrn.com/abstract=2524105.

Pálvölgyi, D., Venter, G., 2015b. On Equilibrium Uniqueness in Multi-Asset Noisy Rational Expectations Economies. Working paper, Available at SSRN: https://ssrn.com/abstract=2631627.

Rahi, R., Zigrand, J.-P., 2018. Information acquisition, price informativeness, and welfare. J. Econom. Theory 177, 558–593.

Rostek, M., Weretka, M., 2012. Pricing inference in small markets. Econometrica 80, 687–711.

Vives, X., 2014. On the possibility of informationally efficient markets. J. Eur. Econom. Assoc. 12, 1200–1239.