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Some properties of the optimal investment strategy in a behavioral portfolio choice model

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Abstract

The aim of this manuscript is to analyze the monotonicity and limit properties of the optimal investment strategy in a behavioral portfolio choice model under cumulative prospect theory over risk aversion coefficient, loss aversion coefficient, and the market opportunity. We show that the optimal investment strategy is nonincreasing of the loss aversion coefficient, and strictly increasing of the Sharpe ratio for normal distributions. The monotonicity properties over risk aversion coefficient depend on the position of the investor and the goodness of the actual and perceived market. The piecewise-linear utility is also discussed. An interesting finding is that when the excess return follows an elliptical distribution, the optimal investment strategy over small mean for piecewise-power and piecewise-linear utility exhibits different limit behavior.

Keywords Cumulative prospect theory (CPT) · Behavioral portfolio choice (BPC) · Monotonicity properties

1 Introduction

Cumulative prospect theory (CPT) has became a powerful tool to capture investors' psychology in decision-making [7,17]. Several single-period behavioral portfolio choice (BPC) models under CPT have been studied in the literature [2,5,8,9,11]. He and Zhou [5] consider a quite general setting and derive the optimal investment strategy (OIS) for two special cases with a piecewise-linear-utility and zero-relative-wealth. He and Zhou [5] also show that the CPT preference value function is not concave on either the positive or the negative half space, and consequently, it is difficult to present a closed-form solution for BPC models in general.

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Because of the non-concavity of CPT preference value functions, most of the literature aims to analyze the properties of OISs or obtain semi-closed OISs. For instance, Bernard and Ghossoub [2] consider a no-shorting restriction and probability distortion, where the reference point corresponds to the terminal wealth when investing the entire initial wealth in the risk-free asset, and show that the OIS is a function of a generalized *Omega measure* of the distribution of the excess return on the risky asset over the risk-free rate. Pirvu and Schulze [11] investigate a model with multiple risky assets which follow a joint elliptical distribution, and obtain a semi-closed OIS by transferring the multi-dimensional optimization problem into a one-dimensional optimization problem. Minsuk and Pirvu [10] consider a model with multiple risky assets which follow a Skewed t distribution, and obtain a semi-closed OIS by reducing the multi-dimensional optimization problem into two one-dimensional optimization problems. Lou et al. [9] try to identify the investment direction, i.e., one investor should optimally long or short the stock. Specifically, Lou et al. [9] show that for two-point distributions, when an investor is in a gain position, the investment direction depends only on whether the mean of the excess return is positive or negative. However, when an investor is in a loss position, the investment direction no longer depends on the actual market opportunity, but the perceived market opportunity, which is jointly described by this investor's risk aversion coefficient and the market condition. In addition, Lou et al. [9] also show that one CPT-investor will optimally long (short) the stock when the excess return follows an elliptical distribution and has a positive (negative) mean no matter whether this investor is in a gain or loss position. Following [9], Lou [8] continues to identify the investment direction for general distributions and show that the result about two-point distributions in the case of gain positions still holds for general distributions when the CPT-investor is sufficiently loss-averse, but no longer holds in the case of loss positions by constructing counterexamples.

Besides single-period models, multi-period and continuous-time BPC models under CPT are also studied in the literature [3,4,6,14–16]. Jin and Zhou [6] formulate a general continuous-time BPC model in a complete market. In the same vein as [5], Carassus and Rasonyi [3] establish easily verifiable and interpretable conditions for the well-posedness of an incomplete discrete-time multi-period BPC model. Shi et al. [14] consider discrete-time multi-period BPC models and derive semi-analytical optimal policies where the reference point is fixed as a constant. Shi et al. [15] propose a partially updated reference point formation rule and obtain a semi-analytical solution. Deng and Pirvu [4] consider a discrete-time multi-period BPC model with a portfolio constraint. Strub and Li [16] analyze and compare the OISs for a multi-period discrete-time BPC problem with loss-aversion and time-varying reference points for different reference point updating rules.

Due to the difficulty of deriving explicit OISs pointed out by [5], in this paper we aim to characterize the monotonicity and limit properties of the OIS of the single-period BPC model in [9] over the model parameters, for instance, the risk aversion coefficient, loss aversion coefficient and the market opportunity. Lou et al. [9] establish the piecewise-linear structure of the OIS. To be specific, it reveals that the OIS is piecewise linear with respect to the relative wealth with vertex at zero. The piecewise-linearity coefficient, which depends only on the risk aversion coefficient, the loss aversion coefficient, and the market opportunity, takes two different values depending on the sign of the relative wealth. Because of the piecewise-linear structure, the investigation of the properties of OISs is equivalent to analyze the properties of the two piecewise-linearity coefficients.

To the best of our knowledge, there is no existing work that analyzes the monotonicity and limit properties of OISs under general settings, even though He and Zhou [5] present some monotonicity analysis of OISs over loss aversion coefficient for piecewise-linear utility functions. Knowing the monotonicity and limit properties of OISs is important because it can help CPT-investors better understand their positions and make better investment decisions under different market circumstances. An interesting finding of this article is that when the risky asset follows an elliptical distribution, the optimal investment strategy over small mean of the excess return exhibits different limit behavior. This finding demonstrates that piecewise-power utility and piecewise-linear utility are two qualitatively different utility functions to describe and characterize CPT-investors' decision-making.

2 The behavioral portfolio choice model

Consider the single-period behavioral portfolio choice model studied by He and Zhou [5] and Lou et al. [9]. The financial market under consideration consists of one risk-free account and one risky asset (stock) with stochastic return $\tilde{R}$. Throughout this paper, we assume that the risk-free asset does not generate interests for simplicity. Let W_0 be the initial wealth of an individual investor at the beginning of this time period, and θ the amount that she invests in the stock. Then, the balance $W_0 - \theta$ goes to the risk-free account. The investor's total wealth at the end of this period can be expressed as $W_0 + (\tilde{R} - 1)\theta$. In our study, shorting is allowed in this market. We define $R := \tilde{R} - 1$ as the excess return which follows a cumulative distribution function $F(\cdot)$ with the no-arbitrage assumption of $0 < \mathbb{P}\big(R < 0\big) < 1$ and $0 < \mathbb{P}\big(R > 0\big) < 1$.

We assume that the investor in the market has CPT preferences. We denote her reference point by B, which serves as a base point to separate gains from losses at the end of the investment period, and express her S-shaped piecewise-power utility function as suggested in [17],

$$u(x) = \begin{cases} x^{\alpha}, & x \geq 0; \\ -K(-x)^{\alpha}, & x < 0, \end{cases} \tag{1}$$

where $K > 1$ is the loss aversion coefficient and $0 < \alpha < 1$ is the risk aversion coefficient.[1] Here we assume that there is no probability distortion for analytical simplicity.

Let $\bar{B} = W_0 - B$ denote the relative wealth of the investor. The CPT preference value, defined as

$$\mathrm{E}[u(W_0 + R\theta)] =: V(\theta),$$

[1] In Sect. 4 we will consider the piecewise-linear utility, i.e., $\alpha = 1$.

of the terminal wealth $W_0 + R\theta$ is given by

$$V(\theta) = \int_{-\frac{\bar{B}}{\theta}}^{+\infty} (\theta t + \bar{B})^{\alpha} dF(t) - K \int_{-\infty}^{-\frac{\bar{B}}{\theta}} (-\theta t - \bar{B})^{\alpha} dF(t) \text{ for } \theta > 0, \tag{2}$$

$$V(\theta) = \int_{-\infty}^{-\frac{\bar{B}}{\theta}} (\theta t + \bar{B})^{\alpha} dF(t) - K \int_{-\frac{\bar{B}}{\theta}}^{+\infty} (-\theta t - \bar{B})^{\alpha} dF(t) \text{ for } \theta < 0, \tag{3}$$

and

$$V(0) = \begin{cases} \bar{B}^{\alpha}, & \bar{B} \geq 0; \\ -K|\bar{B}|^{\alpha}, & \bar{B} < 0. \end{cases}$$

The investor tries to find the OIS θ^* which solves the following optimization problem

$$V(\theta^*) = \max_{\theta \in \mathbb{R}} V(\theta). \tag{4}$$

We make the following three assumptions throughout this paper where the only exception is that we will also consider two-point distributions when analyzing the monotonicity property of the OIS over risk aversion coefficient in the next section.

Assumption 1 Suppose that the cumulative distribution function $F(\cdot)$ of the excess return R is absolutely continuous with a probability density function $f(\cdot)$ which satisfies that $f(t) = O(|t|^{-2-\epsilon})$ for sufficiently large $|t|$,[2] where $\epsilon > 0$.

Assumption 2 Suppose the following well-posedness condition holds,

$$K > K_0, \;\; K_0 \triangleq \max \left\{ \frac{\int_0^{+\infty} t^{\alpha} dF(t)}{\int_{-\infty}^{0} |t|^{\alpha} dF(t)}, \frac{\int_{-\infty}^{0} |t|^{\alpha} dF(t)}{\int_0^{+\infty} t^{\alpha} dF(t)} \right\}. \tag{5}$$

Assumption 3 Suppose that the optimal solution θ^* of (4) is unique.

Assumption 1 guarantees that the CPT preference value function $V(\cdot)$ takes a finite value for any $\theta \in \mathbb{R}$ (see Proposition 1 in [5]). Serval common distributions, e.g., normal, lognormal and logistic distributions, satisfy Assumption 1. As shown in Corollary 1 in [5], Assumption 2 ensures that the CPT model under consideration is well-posed, that is, the optimal solution θ^* is finite. Moreover, Assumption 3 is not restrictive. For instance, our simulation shows that Assumption 3 is always satisfied when the excess return follows a normal distribution with a nonzero mean.

The following theorem establishes the piecewise-linear structure of the OIS of (4) (see Theorem 1 in [9]).

[2] For two functions g and h, $g(t) = O(h(t))$ represents that $\limsup_{t\to\infty} \frac{g(t)}{h(t)} \leq M$ for some constant $M > 0$, while $g(t) = o(h(t))$ represents that $\lim_{t\to\infty} \frac{g(t)}{h(t)} = 0$.

Theorem 1 (Piecewise Linear OIS) *Consider the behavioral portfolio choice problem* (4). *When the relative wealth* $\bar{B} = W_0 - B = 0$, $\arg\max_{\theta\in\mathbb{R}} V(\theta) = 0$, *and when* $\bar{B} \neq 0$, *there exists* γ^*, *which depends only on* α, K, F, *and the sign of* $\bar{B}$ *(but not the absolute value of* $\bar{B}$*), such that the OIS* θ^* *takes the form of*

$$\theta^* = \arg\max_{\theta\in\mathbb{R}} V(\theta) = \gamma^* \bar{B}.$$

The piecewise-linearity result provides great convenience for solving OISs when the investor's reference point varies, because in order to obtain the OISs corresponding to different reference points, it suffices to solve (4) only twice for the two cases of $\bar{B} = 1$ and $\bar{B} = -1$, respectively. Denoting the resulting two optimal solutions θ^* by

$$\gamma_+, \quad -\gamma_-$$

respectively (i.e., $\gamma^* = \gamma_+$ in the case of $\bar{B} = 1$ and $\gamma^* = \gamma_-$ in the case of $\bar{B} = -1$), then the OIS must be $\gamma_+ \bar{B}$ for any positive $\bar{B}$ and $\gamma_- \bar{B}$ for any negative $\bar{B}$.

3 Main results

Because we are unable to obtain an explicit form of $\gamma_\pm$ for general distributions of excess returns,[3] we will explore in this paper the monotonicity and limit properties of $\gamma_\pm$ over the model parameters such as risk aversion coefficient, loss aversion coefficient, and the market opportunity in terms of the mean of the excess return. In the rest of this paper, including all the proofs, the notation $\gamma_+(\cdot)$ indicates a function of some specified variable(s) while keeping all other parameters fixed. Similar notations are also used for $\gamma_-(\cdot)$, $\theta^*(\cdot)$, $V(\cdot)$, $f(\cdot)$ and $F(\cdot)$.

Firstly, although it is not easy to characterize how γ_+ and γ_- vary as a function of the risk aversion coefficient α for general distributions, we do find their relationship for two-point distributions. We use $T(a_0; p_0; a_1; p_1)$ to denote a two-point distribution, which takes two values a_0, a_1 with respective positive probability p_0, p_1, where $a_0 < 0 < a_1$, and $p_0 + p_1 = 1$. Theorem 1 in [9] presents an explicit form of γ_+ and γ_- in the presence of two-point distributions:

$$\gamma_+ = \frac{1 - \dfrac{\frac{a_1}{|a_0|}+1}{\left(\frac{a_1 p_1}{|a_0| p_0}\right)^{\frac{1}{1-\alpha}} + \frac{a_1}{|a_0|}}}{|a_0|}, \quad \text{if } \; a_1 p_1 \neq |a_0| p_0;$$

[3] Corollary 2 in [5] shows that the CPT preference value function $V(\cdot)$ is nonconcave on either $\mathbb{R}^+$ or $\mathbb{R}^-$. The nonconcavity of CPT preference value function brings greater difficulty in solving the optimization problem (4) analytically (see the statements in Section 5 in [5]).

$$\gamma_- = \begin{cases} \dfrac{\dfrac{\frac{|a_0|}{a_1}+1}{\left(K\frac{|a_0|p_0}{a_1p_1}\right)^{\frac{1}{1-\alpha}}-\frac{|a_0|}{a_1}}+1}{-a_1}, & \text{if } a_1^\alpha p_1 > |a_0|^\alpha p_0; \\ \dfrac{\dfrac{\frac{a_1}{|a_0|}+1}{\left(K\frac{a_1p_1}{|a_0|p_0}\right)^{\frac{1}{1-\alpha}}-\frac{a_1}{|a_0|}}+1}{|a_0|}, & \text{if } a_1^\alpha p_1 < |a_0|^\alpha p_0. \end{cases} \tag{6}$$

Proposition 1 *Suppose the excess return* $R \sim T(a_0, p_0; a_1, p_1)$ *and* $K > \sup\left\{\frac{a_1^\alpha p_1}{|a_0|^\alpha p_0}, \frac{|a_0|^\alpha p_0}{a_1^\alpha p_1}, 0 < \alpha < 1\right\}$. *Then we have*

(i) $\gamma_+(\cdot)$ *is strictly increasing (decreasing) of* α *on* $(0, 1)$ *if* $a_1 p_1 > |a_0| p_0$ *(*$a_1 p_1 < |a_0| p_0$*).*

(ii) $\gamma_-(\cdot)$ *is strictly increasing (decreasing) of* α *on* $(0, 1)$ *if* $a_1^\alpha p_1 > |a_0|^\alpha p_0$ *for any* $0 < \alpha < 1$ *(*$a_1^\alpha p_1 < |a_0|^\alpha p_0$ *for any* $0 < \alpha < 1$*).*

Proof This is straightforward from the expression (6) for two-point distributions. □

Proposition 1 shows that the monotonicity properties over risk aversion coefficient for two-point distributions depend on the position of the investor and the goodness of the actual and perceived market. When the investor is in a gain position, the monotonicity property depends on whether the actual market is good, i.e., the mean of the excess return is positive or negative. When an investor is in a loss position, the monotonicity property depends on the perceived market opportunity, which is jointly described by the investor's risk aversion coefficient and the market condition.

In decision theory, the scalar $-\frac{xU''(x)}{U'(x)}$ is defined as a measure of the relative risk aversion of the utility function U. A well-known result in [1,12] indicates that in a single-period market (with a positive mean of the excess return) consisting of one risky asset and one risk-free asset, in which investors maximize their expected utility, a more risk averse investor (i.e., with a larger relative risk aversion) will invest a lower proportion of the portfolio in the risky asset. In our setting with a two-point distribution where the utility is represented by the one in (1) and the relative wealth is positive, our monotonicity result (the first part (i) of Proposition 1) reveals that an investor with larger α should hold a higher proportion of the portfolio in the risky asset. Indeed, the "relative risk aversion" of the piecewise-power utility u equals $1-\alpha$ except at the point 0, which implies that the investor with larger α is less risk averse or more risk seeking. Our monotonicity result under the framework of CPT for gain-position investors is thus consistent with that under Expected Utility Theory (EUT) given in [1,12].

Secondly, we consider the monotonicity properties of γ_+ and γ_- over loss aversion coefficient K.

Proposition 2 *Suppose that the optimal solution of* (4) *is positive for both cases of* $\bar{B} > 0$ *and* $\bar{B} < 0$ *(i.e.,* $\gamma_+ > 0, \gamma_- < 0$*).*

(i) *Suppose* $V(\cdot)$ *is nondecreasing on* $(0, \theta^*]$. *Then* $\gamma_+(\cdot)$ *is nonincreasing of* K *on* $(K_0, +\infty)$. *If additionally, the excess return* R *is unbounded from below (i.e.,* $F(t) > 0$ *for any* $t < 0$*), then* $\gamma_+(\cdot)$ *is strictly decreasing of* K *on* $(K_0, +\infty)$.

(ii) *Suppose* $V(\cdot)$ *is nondecreasing on* $(0, \theta^*]$. *Then* $\gamma_-(\cdot)$ *is nondecreasing of* K *on* $(K_0, +\infty)$. *If additionally, either the excess return* R *is unbounded from above (i.e.,* $F(t) < 1$ *for any* $t > 0$*), or the mean of* R *is positive, then* $\gamma_-(\cdot)$ *is strictly increasing of* K *on* $(K_0, +\infty)$.

Proof (i). Let $\bar{B} = 1$. For any K_1, K_2 with $K_0 < K_1 < K_2$, from (2) we have that for any $\theta > 0$,

$$\frac{\partial V(\theta, K_1)}{\partial \theta} \geq \frac{\partial V(\theta, K_2)}{\partial \theta}.$$

Combing the preceding relation with the nondecreasingness of $V(\cdot)$ on $(0, \theta^*]$, we obtain that $\gamma_+(\cdot)$ is nondecreasing of K on $(K_0, +\infty)$. Moreover, if R is unbounded from below, the second inequality becomes strict and consequently, the strict increasingness holds.

(ii). Let $\bar{B} = -1$. By the first-order condition, the optimal solution $\theta^*(K_1)$, which is positive by hypothesis, satisfies $\frac{\partial V(\theta^*(K_1), K_1)}{\partial \theta} = 0$, i.e.,

$$\int_{\frac{1}{\theta^*(K_1)}}^{+\infty} [\theta^*(K_1)t - 1]^{\alpha-1} t dF(t) + K_1 \int_{-\infty}^{\frac{1}{\theta^*(K_1)}} [-\theta^*(K_1)t + 1]^{\alpha-1} t dF(t) = 0.$$

As a result,

$$\int_{-\infty}^{\frac{1}{\theta^*(K_1)}} [-\theta^*(K_1)t + 1]^{\alpha-1} t dF(t) \leq 0,$$

and so when $K_2 > K_1$,

$$\int_{\frac{1}{\theta^*(K_1)}}^{+\infty} [\theta^*(K_1)t - 1]^{\alpha-1} t dF(t) + K_2 \int_{-\infty}^{\frac{1}{\theta^*(K_1)}} [-\theta^*(K_1)t + 1]^{\alpha-1} t dF(t) \leq 0,$$

i.e., $\frac{\partial V(\theta^*(K_1), K_2)}{\partial \theta} \leq 0$. By the monotonicity assumption of $V(\cdot, K_2)$ on $(0, \theta^*(K_2)]$, we have $\theta^*(K_2) \leq \theta^*(K_1)$. Therefore, $\theta^*(\cdot)$ is nonincreasing and consequently, $\gamma_-(\cdot)$ is nondecreasing of K on $(K_0, +\infty)$.

When R is unbounded from above,

$$\int_{\frac{1}{\theta^*(K_1)}}^{+\infty} [\theta^*(K_1)t - 1]^{\alpha-1} t dF(t) > 0,$$

and as a result,

$$\int_{-\infty}^{\frac{1}{\theta^*(K_1)}} [-\theta^*(K_1)t + 1]^{\alpha-1} t dF(t) < 0.$$

Note that the inequality in the preceding relation is strict and hence the monotonicity of $\theta^*(\cdot)$ is strict. When $\mathrm{E}[R] > 0$, it also holds that

$$\int_{\frac{1}{\theta^*(K_1)}}^{+\infty} [\theta^*(K_1)t - 1]^{\alpha-1} t dF(t) > 0$$

and then the monotonicity follows. In fact, otherwise, the excess return R is bounded from above, i.e., $R \leq \frac{1}{\theta^*(K_1)} = \frac{1}{|\gamma_-|}$ with probability one. Consequently, by Lemma 2 in [9],[4] we have $|\gamma_-| \times \frac{1}{|\gamma_-|} > 1$, raising a contradiction. □

Corollary 1 in [9] shows that when the excess return R follows an elliptical distribution (see the definition later), the optimal solution of (4) is positive when the mean of R is positive no matter $\bar{B} > 0$ or $\bar{B} < 0$. However, for general distributions it is difficult to identify the relationship between the positiveness/negativeness of the optimal solution of (4) and the market opportunity.

The monotonicity result on loss aversion coefficient K has been revealed in Theorem 5 of [5] for piecewise-linear utilities. The monotonicity results in Proposition 2 reveal that when an investor is more loss averse, she should hold lower proportion of the portfolio in the risky asset to decrease the risk of loss caused by the risky asset. This is intuitive.

A random variable X is said to follow an elliptical distribution $E(\mu, \sigma, g)$ with $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}_+$, if its probability density function has the form:

$$f(t) = bg\Big(\frac{(t-\mu)^2}{\sigma^2}\Big),$$

where $b > 0$ is a constant, $g : [0, +\infty) \to [0, +\infty)$ is a Lebesgue measurable, nonnegative, strictly decreasing function, which is called density generator. The class of elliptical distributions includes many well-known distributions, e.g., normal distributions ($g(t) = e^{-\frac{1}{2}t}$), Student's t distributions ($g(t) = (1 + \frac{t}{\nu})^{-\frac{1+\nu}{2}}$, $\nu \geq 1$ is an integer), and logistic distributions ($g(t) = \frac{e^{\sqrt{t}}}{(1+e^{\sqrt{t}})^2}$).

In the following discussions of this section, we assume that the excess return R follows an elliptical distribution with a finite and positive mean μ, finite variance, which coincides with σ^2 up to a constant. Corollary 1 in [9] informs us that $\gamma_+ > 0$ and $\gamma_- < 0$. Note that for an elliptical distribution with a positive mean,

$$\frac{\int_0^{+\infty} t^\alpha dF(t)}{\int_{-\infty}^0 |t|^\alpha dF(t)} > 1,$$

and the piecewise-linearity coefficients γ_+ and γ_- depend on the parameters μ and σ only through the Sharpe ratio μ/σ [13]. So the results about the piecewise-linearity coefficients in the following proposition also apply to the situation that σ is increasing (to positive infinity) when keeping μ fixed.

[4] Specifically, it shows that if the excess return R has a positive mean and is bounded from above, i.e., $\mathrm{E}[R] > 0$ and $R \leq M$ with probability one for some $M > 0$, then $|\gamma_-|M > 1$.

Proposition 3 *Suppose that the excess return* $R \sim E(\mu, \sigma, g)$*,* $\mu > 0$*, and*

$$K > \sup_{0<\mu\leq\bar{\mu}} \frac{\int_0^{+\infty} t^\alpha dF(t,\mu)}{\int_{-\infty}^0 |t|^\alpha dF(t,\mu)}$$

for some $\bar{\mu} > 0$*. Then we have*

(i) $\lim_{0<\mu\to 0} \gamma_+(\mu) = 0$*;*
(ii) $\lim_{0<\mu\to 0} \gamma_-(\mu) < 0$ *if* $f(t) = o(|t|^{-3})$ *for sufficiently large* $|t|$*;*
(iii) *If the excess return R follows a normal distribution* $N(\mu,\sigma)$ *and* $V(\cdot)$ *is strictly decreasing of* θ *on* $[\theta^*, +\infty)$*, then* $\gamma_+(\cdot)$ *is strictly increasing of* μ*, and* $\gamma_-(\cdot)$ *is strictly decreasing of* μ *on* $(0, \bar{\mu}]$*;*
(iv) *The maximum CPT preference value* $\max_{\theta\in\mathbb{R}} V(\theta,\cdot)$ *is strictly increasing of* μ *on* $(0, \bar{\mu}]$.

Proof Let $V(\theta,\mu)$ be the CPT preference value function with the emphasis that it is a function of the mean μ of the excess return.

(i) We show the conclusion by contradiction. Hence suppose $\limsup_{0<\mu\to 0} \gamma_+(\mu) > 0$. Then there exist $w > 0$ and infinitely many sufficiently small $\mu > 0$ such that the optimal solution of (4) with $\bar{B} = 1$ is not less than w. Note that the CPT preference value of the optimal solution is not less than $\bar{B}^\alpha = 1$ (because it is not less than the CPT preference value at zero). However, when $\mu = 0$, for any $\theta > 0$,

$$\begin{aligned}
\frac{V'(\theta,0)}{\alpha} &= \int_{-\frac{1}{\theta}}^{+\infty} (\theta t+1)^{\alpha-1} t f(t,0)dt + K\int_{-\infty}^{-\frac{1}{\theta}} (-\theta t-1)^{\alpha-1} t f(t,0)dt \\
&< \int_{-\frac{1}{\theta}}^{+\infty} (\theta t+1)^{\alpha-1} t f(t,0)dt + \int_{-\infty}^{-\frac{1}{\theta}} (-\theta t-1)^{\alpha-1} t f(t,0)dt \\
&= \int_0^{+\infty} \left[(\theta t+1)^{\alpha-1} - |\theta t-1|^{\alpha-1}\right] t f(t,0)dt \\
&< 0,
\end{aligned}$$

where the first inequality follows from the fact $K > 1$ and the second inequality from the two relations $(\theta t+1)^{\alpha-1} < |\theta t-1|^{\alpha-1}$ and $f(t,0) = f(-t,0)$. Therefore, for any $\theta \geq w$,

$$V(\theta,0) \leq V(w,0) \leq V\Big(\frac{w}{2},0\Big) - \frac{\varsigma w}{2} \leq V(0,0) - \frac{\varsigma w}{2} = 1 - \frac{\varsigma\alpha w}{2},$$

where

$$\varsigma = \inf_{\frac{w}{2}\leq\theta\leq w} \int_0^{+\infty} |(\theta t+1)^{\alpha-1} - |\theta t-1|^{\alpha-1}| t f(t,0)dt > 0.$$

This along with the continuity of $V(\theta,\mu)$ in μ leads to a contradiction. Thus, $\lim_{0<\mu\to 0} \gamma_+(\mu) = 0$.

(ii) It suffices to show that when $\bar{B} = -1$,

$$\max_{\theta \in [0,+\infty)} V(\theta, 0) > V(0, 0) = -K.$$

Note that for $\theta > 0$,

$$\begin{aligned}
\frac{V'(\theta, 0)}{\alpha} &= \int_{\frac{1}{\theta}}^{+\infty} (\theta t - 1)^{\alpha-1} t f(t, 0) dt + K \int_{-\infty}^{\frac{1}{\theta}} (-\theta t + 1)^{\alpha-1} t f(t, 0) dt \\
&= \int_{\frac{1}{\theta}}^{\frac{2}{\theta}} (\theta t - 1)^{\alpha-1} t f(t, 0) dt + K \int_{\frac{1}{2\theta}}^{\frac{1}{\theta}} (-\theta t + 1)^{\alpha-1} t f(t, 0) dt \\
&\quad + \int_{\frac{2}{\theta}}^{+\infty} (\theta t - 1)^{\alpha-1} t f(t, 0) dt + K \int_{-\infty}^{\frac{1}{2\theta}} (-\theta t + 1)^{\alpha-1} t f(t, 0) dt \\
&=: z_1(\theta) + z_2(\theta) + z_3(\theta) + z_4(\theta).
\end{aligned}$$

Some simple computations give

$$\begin{aligned}
&z_1(\theta) + z_2(\theta) = O\Big(\frac{1}{\theta^2} f\big(\frac{1}{\theta}, 0\big)\Big), \\
&z_3'(\theta) + z_4'(\theta) = \frac{4}{\theta^3} f\big(\frac{2}{\theta}, 0\big) + (\alpha - 1) \int_{\frac{2}{\theta}}^{+\infty} (\theta t - 1)^{\alpha-2} t^2 f(t, 0) dt \\
&\quad + K\Big[- \frac{1}{2^{1+\alpha}\theta^3} f\big(\frac{1}{2\theta}, 0\big) + (1 - \alpha) \int_{-\infty}^{\frac{1}{2\theta}} (-\theta t + 1)^{\alpha-2} t^2 f(t, 0) dt \Big].
\end{aligned}$$

Observe that for any $M > 0$ and $\theta > 0$,

$$\begin{aligned}
\int_{-\infty}^{\frac{1}{2\theta}} (-\theta t + 1)^{\alpha-2} t^2 f(t, 0) dt &\geq \int_{-M}^{\frac{1}{2\theta}} (-\theta t + 1)^{\alpha-2} t^2 f(t, 0) dt \\
&\geq (\theta M + 1)^{\alpha-2} \int_{-M}^{\frac{1}{2\theta}} t^2 f(t, 0) dt,
\end{aligned}$$

where the second inequality follows from the two relations $(-\theta t + 1)^{\alpha-2} > 1$ for $1/(2\theta) \leq t \leq 0$ and $(-\theta t + 1)^{\alpha-2} > (\theta M + 1)^{\alpha-2}$ for $-M \leq t \leq 0$. Therefore,

$$\lim_{0<\theta\to 0} \int_{-\infty}^{\frac{1}{2\theta}} (-\theta t + 1)^{\alpha-2} t^2 f(t, 0) dt \geq \int_{-M}^{+\infty} t^2 f(t, 0) dt$$

for any $M > 0$, from which we have

$$\lim_{0<\theta\to 0} \int_{-\infty}^{\frac{1}{2\theta}} (-\theta t + 1)^{\alpha-2} t^2 f(t, 0) dt \geq \int_{-\infty}^{+\infty} t^2 f(t, 0) dt > 0.$$

Furthermore,

$$\lim_{0<\theta\to 0}\int_{\frac{2}{\theta}}^{+\infty}(\theta t-1)^{\alpha-2}t^2 f(t,0)dt=0.$$

The preceding two relations and the hypothesis $f(t)=o(|t|^{-3})$ imply that $V'(\theta,0)>0$ for all positive but sufficiently small θ. The conclusion (ii) follows.

(iii) We see that for $\theta>0$,

$$\frac{\partial V(\theta,\mu)}{\partial\theta}=\alpha\Big[\int_{-\frac{\bar B}{\theta}}^{+\infty}(\theta t+\bar B)^{\alpha-1}tf(t,\mu)dt + K\int_{-\infty}^{-\frac{\bar B}{\theta}}(-\theta t-\bar B)^{\alpha-1}tf(t,\mu)dt\Big], \tag{7}$$

$$\frac{\partial^2 V(\theta,\mu)}{\partial\mu\partial\theta}=\alpha\Big[\int_{-\frac{\bar B}{\theta}}^{+\infty}(\theta t+\bar B)^{\alpha-1}t(t-\mu)f(t,\mu)dt + K\int_{-\infty}^{-\frac{\bar B}{\theta}}(-\theta t-\bar B)^{\alpha-1}t(t-\mu)f(t,\mu)dt\Big]. \tag{8}$$

Since $V(\cdot,\mu)$ achieves its maximum at $\theta^*(\mu)$, which is positive, $\frac{\partial V(\theta^*(\mu),\mu)}{\partial\theta}=0$. This together with (7) and (8) yields

$$\begin{aligned}\frac{\partial^2 V(\theta^*(\mu),\mu)}{\partial\mu\partial\theta}&=\alpha\Big(\int_{-\frac{\bar B}{\theta^*(\mu)}}^{+\infty}[\theta^*(\mu)t+\bar B]^{\alpha-1}t^2 f(t,\mu)dt\\&\quad+K\int_{-\infty}^{-\frac{\bar B}{\theta^*(\mu)}}[-\theta^*(\mu)t-\bar B]^{\alpha-1}t^2 f(t,\mu)dt\Big)\\&>0.\end{aligned}$$

As a result,

$$\frac{\partial V(\theta^*(\mu),\mu+\epsilon)}{\partial\theta}>\frac{\partial V(\theta^*(\mu),\mu)}{\partial\theta}=0$$

when $\epsilon>0$ is sufficiently small. The preceding inequality and the monotonicity assumption of $V(\cdot,\mu)$ on $[\theta^*(\mu),+\infty)$ imply that $\theta^*(\mu+\epsilon)>\theta^*(\mu)$ for all sufficiently small $\epsilon>0$. Thus, $\gamma_+(\cdot)$ is strictly increasing and $\gamma_-(\cdot)$ is strictly decreasing of μ on $(0,\bar\mu]$.

(iv) Let $0 < \nu < \mu \leq \bar{\mu}$. We have for any $\theta > 0$,

$$\begin{aligned}
&V(\theta, \mu, W_0) \\
&= \int_{-\frac{W_0-B}{\theta}}^{+\infty} (\theta t + W_0 - B)^\alpha f(t,\mu)dt - K \int_{-\infty}^{-\frac{W_0-B}{\theta}} (-\theta t - W_0 + B)^\alpha f(t,\mu)dt \\
&= \int_{-\frac{W_0+\theta(\mu-\nu)-B}{\theta}}^{+\infty} [\theta s + W_0 + \theta(\mu-\nu) - B]^\alpha f(s,\nu)ds \\
&\quad - K \int_{-\infty}^{-\frac{W_0+\theta(\mu-\nu)-B}{\theta}} [-\theta s - W_0 - \theta(\mu-\nu) + B]^\alpha f(s,\nu)ds \\
&= V(\theta, \nu, W_0 + \theta(\mu-\nu)).
\end{aligned}$$

Equivalently, the CPT preference value of investing θ in the risky asset with initial wealth W_0 and mean μ equals that with a smaller mean ν, but more initial wealth $W_0 + \theta(\mu - \nu)$. We also have for any fixed $\theta > 0$,

$$\frac{\partial V(\theta, \nu, W_0)}{\partial W_0} \geq \alpha \int_{-\infty}^{+\infty} |\theta t + W_0 - B|^{\alpha-1} f(t,\nu)dt > 0,$$

which leads to

$$V(\theta, \mu, W_0) = V(\theta, \nu, W_0 + \theta(\mu-\nu)) > V(\theta, \nu, W_0)$$

due to $W_0 + \theta(\mu - \nu) > W_0$. The conclusion (iv) follows. □

It is easy to verify that normal, logistic and Student's t distributions with degree of freedom $\nu \geq 3$ satisfy the hypothesis in Proposition 3 (ii). The monotonicity results in Proposition 3 (iii) and (iv) reveal an intuitive fact that better the actual market in terms of the mean μ, the higher proportion of the portfolio invested in the risky asset and the larger CPT preference value. Proposition 3 (i) and (ii) show that the limits of the OIS for small mean of the excess return for the two cases of positive and negative relative wealth are different. When one investor is in a gain position, intuitively she is risk averse on the whole and then will invest smaller and smaller amount in the risky asset to decrease the risk of loss caused by the risky asset as the market becomes more neutral (in the sense that the mean of the excess return gets closer and closer to zero). Whereas when one investor is in a loss position, intuitively she is risk seeking on the whole and consequently, will invest a certain amount in the risky asset to "break even" even though the market becomes more neutral.

4 Piecewise-linear utility

Here we present some discussions on the piecewise-linear utility (the utility in (1) with $\alpha = 1$), i.e., $u(x) = x$ for $x \geq 0$, $u(x) = Kx$ for $x < 0$. With this special utility, the well-posedness condition (5) simplifies to

$$K > \max\left\{\frac{\int_0^{+\infty} t dF(t)}{\int_{-\infty}^0 |t| dF(t)}, \frac{\int_{-\infty}^0 |t| dF(t)}{\int_0^{+\infty} t dF(t)}\right\}.$$

We assume that the actual market is good, i.e., the mean of the excess return is positive ($\mathrm{E}[R] > 0$). It follows from (3) that for $\theta < 0$,

$$V'(\theta) = \int_{-\infty}^{-\frac{\bar{B}}{\theta}} tf(t)dt + K \int_{-\frac{\bar{B}}{\theta}}^{+\infty} tf(t)dt.$$

We can see that $V'(\cdot)$ is nonincreasing of θ on $(-\infty, 0)$ no matter the relative wealth $\bar{B} > 0$ or $\bar{B} < 0$,

$$\lim_{0>\theta\to 0} V'(\theta) = \mathrm{E}[R] > 0$$

in the presence of $\bar{B} > 0$, and

$$\lim_{0>\theta\to 0} V'(\theta) = K\mathrm{E}[R] > 0$$

in the presence of $\bar{B} < 0$. Consequently, the optimal solution of (4) must be positive. Note that when the utility is piecewise linear, the utility function is concave. The above analysis reveals that the optimal solution of (4) is positive when the actual market is good with a positive mean of the excess return no matter the relative wealth $\bar{B}$ is positive or negative. This is consistent with the classical result in EUT. He and Zhou [5] derive the OIS for the piecewise-linear utility based on an additional assumption that the CPT preference value function satisfies that $V(\theta) \leq V(0)$ for any $\theta < 0$. In fact, our analysis shows that this additional assumption is not necessary.

We also observe that for $\theta > 0$,

$$V'(\theta) = \int_{-\frac{\bar{B}}{\theta}}^{+\infty} tf(t)dt + K \int_{-\infty}^{-\frac{\bar{B}}{\theta}} tf(t)dt.$$

It can be seen that $V'(\cdot)$ is nonincreasing of θ on $(0, +\infty)$ no matter $\bar{B} > 0$ or $\bar{B} < 0$. By letting $V'(\theta^*) = 0$, the piecewise-linearity coefficient takes the following form,

$$\gamma_+ = -\frac{1}{v_-}, \ \gamma_- = -\frac{1}{v_+},$$

where v_+, v_- are the roots of

$$\int_v^{+\infty} tf(t)dt + K \int_{-\infty}^{v} tf(t)dt = 0 \tag{9}$$

on $(0, +\infty)$, $(-\infty, 0)$, respectively.

We conclude that if the probability density function $f(\cdot)$ satisfies the property that $f(t) = 0$ for any $t \le t_1$ if $f(t_1) = 0$ for some $t_1 < 0$, then the root of (9) on $(-\infty, 0)$ is unique. The uniqueness follows immediately if the excess return R is unbounded from below, i.e., $f(t) > 0$ for any $t < 0$. In fact, the uniqueness also holds if the excess return is bounded from below. We show this by contradiction. Suppose v_-^1 and v_-^2 with $v_-^1 < v_-^2 < 0$ are two roots of (9) on $(-\infty, 0)$. Then it follows from (9) and the fact $K > 1$ that $\int_{v_-^1}^{v_-^2} tf(t)dt = 0$, which combines with the previous hypothesis on the probability density function forces that $f(t) = 0$ for any $t \le v_-^2$. Consequently, any value in the interval $\left[-\frac{1}{v_-^1}, -\frac{1}{v_-^2}\right]$ is an optimal solution of (4) in the presence of $\bar{B} = 1$. However, for any $\bar{\theta} \in \left(-\frac{1}{v_-^1}, -\frac{1}{v_-^2}\right)$, $V'(\bar{\theta}) = \mathrm{E}[R] > 0$, raising a contradiction. Similarly, we can also show that if the probability density function $f(\cdot)$ satisfies the property that $f(t) = 0$ for any $t \ge t_2$ if $f(t_1) = 0$ for some $t_2 > 0$, the root of (9) on $(0, +\infty)$ is also unique.

Proposition 4 *Suppose the utility is piecewise linear (i.e., $\alpha = 1$). Then we have*

(i) $\gamma_+(\cdot)$ *is strictly decreasing of* K *and* $\gamma_-(\cdot)$ *is strictly increasing of* K *on* $(K_0, +\infty)$*;*

Furthermore, if additionally the excess return R *follows an elliptical distribution* $E(\mu, \sigma, g)$ *with* $\mu > 0$*, and* $K > \sup_{0<\mu\le\bar{\mu}} \frac{\int_0^{+\infty} t^\alpha dF(t,\mu)}{\int_{-\infty}^0 |t|^\alpha dF(t,\mu)}$ *for some* $\bar{\mu} > 0$*, we have*

(ii) $|\gamma_-| > \gamma_+$*;*
(iii) $\gamma_+(\cdot)$ *is strictly increasing of* μ *on* $(0, \bar{\mu}]$*, and* $\lim_{0<\mu\to 0} \gamma_+(\mu) = 0$*;*
(iv) $\gamma_-(\cdot)$ *is strictly decreasing of* μ *on* $(0, \bar{\mu}]$*, and* $\lim_{0<\mu\to 0} \gamma_-(\mu) = 0$.

Proof Take $K_2 > K_1 > K_0$. Let v_+^1 and v_+^2 be two respective solutions of the Eq. (9) on $(0, +\infty)$ corresponding to K_1 and K_2 such that

$$\int_{v_+^i}^{+\infty} tf(t)dt + K_i \int_{-\infty}^{v_+^i} tf(t)dt = 0, \ i = 1, 2.$$

Observing the relation

$$\int_{v_+^1}^{+\infty} tf(t)dt + \int_{-\infty}^{v_+^1} tf(t)dt = \int_{v_+^2}^{+\infty} tf(t)dt + \int_{-\infty}^{v_+^2} tf(t)dt = \mathrm{E}[R] > 0,$$

we have

$$(1 - K_i) \int_{-\infty}^{v_+^i} tf(t)dt = \mathrm{E}[R], \ i = 1, 2.$$

Consequently, $v_+^2 > v_+^1$, implying that $\gamma_-(\cdot)$ is strictly increasing of K on $(K_0, +\infty)$. We can show the monotonicity of $\gamma_+(\cdot)$ by similar arguments. The conclusion (i) follows.

It follows the Eq. (9) that $\int_{v_-}^{v_+} tf(t)dt = 0$. Because the excess return follows an elliptical distribution with a positive mean, $|v_-| > v_+$. Then (ii) follows. The monotonicity results in (iii) and (iv) are straightforward and the limit results in (iii) and (iv) can be shown by contradiction. □

Proposition 4 (i), (iii) and the monotonicity result in (iv) are consistent with the results in Proposition 3. However, Proposition 4 (iv) and Proposition 3 (ii) reveal that the two limits of the coefficient γ_- for small mean of the excess return in the two cases of $0 < \alpha < 1$ and $\alpha = 1$ are different. Different from the piecewise-power utility u in (1) with $0 < \alpha < 1$ which is concave on $\mathbb{R}^+$ and convex on $\mathbb{R}^-$, the piecewise-linear utility with $\alpha = 1$ is concave on the whole real space $\mathbb{R}$. So when one investor's utility is piecewise linear, this investor is risk averse and then will invest smaller and smaller amount in the risky asset as the market becomes more neutral no matter she is in a gain or loss position. This explanation goes along with the limit results in Proposition 4 (iii) and (iv).

5 Conclusion

In this paper, we analyzed the monotonicity and limit properties of the optimal investment strategy in a behavioral portfolio choice model over risk aversion coefficient, loss aversion coefficient, and the market opportunity in terms of the mean of the excess return. The piecewise-linear utility is also discussed. An interesting finding is that when the excess return follows an elliptical distribution, the optimal investment strategy over small mean for piecewise-power and piecewise-linear utility exhibits different limit behavior. This finding demonstrates that piecewise-power utility and piecewise-linear utility are qualitatively different utility functions to describe and characterize CPT-investors' decision-making. The monotonicity and limit properties can help CPT-investors better understand the market circumstance and make better investment decisions. Although this paper attempts to analyze the properties of OISs for general distributions of the risky asset, some analysis is still limited to special distributions. The investigation on the monotonicity and limit properties over risk aversion coefficient and the mean of the excess return for general distributions is challenging and will be studied in our future research.

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