Mathematical Social Sciences 109 (2021) 93–105

Contents lists available at ScienceDirect

Mathematical Social Sciences

journal homepage: www.elsevier.com/locate/mss

The equivalence of two rational expectations equilibrium economies with different approaches to processing neighbors' information[☆]

Youcheng Lou[a,*], Shouyang Wang[a,b,c]

[a] MDIS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, China
[b] Center for Forecasting Science, Chinese Academy of Sciences, China
[c] School of Economics and Management, University of Chinese Academy of Sciences, China

ARTICLE INFO

Article history:
Received 17 January 2020
Received in revised form 4 November 2020
Accepted 22 November 2020
Available online 30 November 2020

Keywords:
Rational expectations equilibrium
Social networks
Network structure
Signal structure

ABSTRACT

We consider a finite-agent Hellwig (1980) economy with an extension to allow traders to observe their neighbors' signals in an exogenously given social network. There are two potential approaches for traders to process observed signals: directly infer information about the fundamentals from the complete collection of observed signals, or indirectly from an average of observed signals. The two approaches lead to different information sets for traders. In this study, we investigate whether the two economies corresponding to the two approaches are equivalent in the sense that they have the same market equilibrium. For general network and signal structures, we present a necessary and sufficient condition for the equivalence, revealing that the two finite-agent economies are not equivalent in general unless the network structure and signal structure coordinate well. When traders have homogeneous preferences and the signal structure takes the classical form in the literature, we find that the two finite-agent economies are equivalent for regular graphs, but not for chain and star graphs. Finally, for the classical signal structure, we show that the two large economies, defined as the limit of a sequence of replica finite-agent economies, are equivalent for any network structure.

1. Introduction

Consider a rational expectations equilibrium (Hellwig, 1980) economy with an extension to an environment of social networks. Each trader initially holds a private signal. Traders are connected through an exogenously given social network in which traders can observe their neighbors' signals.

The most popular and most reasonable approach for traders to process observed signals from their neighbors is to directly infer information about the fundamentals of the risky asset from the *complete collection* of observed signals (as well as the price). However, Ozsoylev and Walden (2011) propose a different approach in a finite-agent model[1]: that traders use a *simple average*, instead of the complete collection, of observed signals to make Bayesian inference. The network structure determines how information diffuses through the network; the information processing approach determines the (new) information available to traders, and traders' information determines their demand for the risky asset. Therefore, the network structure, signal structure, and information processing approach jointly determine the market equilibrium. The two information processing approaches may lead to different market equilibriums. In this study, we address whether the two resulting economies corresponding to the two information processing approaches are *equivalent* in the sense that they have the *same* market equilibrium (the same equilibrium price and optimal demands).

As the signal structure in our model is general, when we investigate the equivalence between the two economies, we consider a sufficient statistic for observed signals (i.e., the conditional mean of the fundamental) instead of a simple average of observed signals.[2] Besides normality and non-degeneracy, we impose no more requirements on the signals.

☆ This work was supported in part by the National Natural Science Foundation of China under Grant 71971208.

* Correspondence to: Academy of Mathematics and Systems Science, Chinese Academy of Sciences, No. 55 Zhongguancun East Road, Beijing 100190, China.
E-mail address: louyoucheng@amss.ac.cn (Y. Lou).

1 It is emphasized that the finite-agent model in Ozsoylev and Walden (2011) only serves as a benchmark to facilitate the later analysis on large economies. In fact, the focus of Ozsoylev and Walden (2011) is on the analysis of the impact of the properties of *large* information networks, *not* finite-agent networks, on assets prices.

2 This generalization from a simple average to a sufficient statistic is reasonable because a sufficient statistic for observed signals reduces to a simple average of observed signals for the classical signal structure being a sum of the fundamental and an independent noise (with identical variance). See also more explanations in Section 2.

https://doi.org/10.1016/j.mathsocsci.2020.11.003
0165-4896/

Motivations. Although averaging signals is a reasonable approach,[3] it is not clear why traders average observed signals from their neighbors rather than infer information from the complete collection of observed signals.[4] Although the average of signals is a sufficient statistic for the complete collection of signals when signals take the classical form of a sum of the fundamental and an independent noise and have the same precision (Ozsoylev and Walden, 2011), it is not the case when signal precisions differ across traders. In addition, even if the average of signals is a sufficient statistic for the complete collection of observed signals, the price and the average of signals together may no longer be a sufficient statistic for the price and the complete collection of observed signals because the price serves as an endogenous public signal determined by market-clearing conditions. In other words, traders may have an incentive to make Bayesian inference based on the complete collection of observed signals instead of the averaged signal because they believe that the complete collection offers more information and thus can provide them with a higher expected utility. Also, as claimed by Lou ⓡ al. (2019): "*Except for special situations, there is no reason why each trader should take an average of signals. Indeed, averaging isn't the problem: any exogenous aggregation method is suspect. [......] A model of signal-sharing in networks is therefore necessary to handle the case of multidimensional signals at the individual level.*" This motivates the current study, wherein we formally justify the above arguments and claim by investigating the equivalence of the two economies.

It is reasonable that traders use a simple average of signals to make Bayesian inference when the two economies are equivalent. However, it is not reasonable when they are not equivalent. Specifically, we show that when the two economies are not equivalent, in the economy using the averaging approach, there exists at least one trader who has an incentive to use the complete collection of observed signals to increase their expected utility, besides the averaged signal (see Proposition 1). This reveals that the economy using an averaging approach is not *"stable"* in some sense and thus validates the claim in Lou ⓡ al. (2019).

Main Results. The main result of the paper is a series of characterizations of the equivalence of the two economies. As a basis for the subsequent development, we first show that the two economies are equivalent if and only if the conditional means under the two information processing approaches are almost surely identical (see Proposition 2). We then show that the two economies are equivalent for *any* signal structure when the network graph is complete (see Theorem 1). We also present a *necessary and sufficient* condition on the solvability of a system of equations for the equivalence of the two economies (see Theorem 2). It shows that the two finite-agent economies are *not* equivalent in general. We show that, for any signal structure (respectively, non-complete graph), there exists a corresponding network graph (respectively, signal structure) such that the two economies are not equivalent (see Theorems 3 and 4). In addition, for tractability, we also revisit the classical signal structure used in Hellwig (1980) and Ozsoylev and Walden (2011) that takes the form of a sum of fundamental and independent noise. We consider homogeneous preferences, and find that the two finite-agent economies are equivalent for regular graphs, but not for chain and star graphs (see Corollary 2). Our results reveal that the two finite-agent economies generally do not have the same market equilibrium unless the network structure and signal structure coordinate well.

We also consider a *large* replica economy similar to that in Han and Yang (2013) and Walden (2019). The large economy is defined as the limit of a sequence of finite-agent economies where each finite-agent economy consists of several disjoint independent subnetworks. These subnetworks have equal network size and identical network structure. When signals take the classical form, we show that the two large replica economies corresponding to the two information processing approaches are equivalent for any network structure (see Theorem 5). Our analysis justifies the assumption in Ozsoylev and Walden (2011) that traders take an average of their neighbors' signals to make Bayesian inference in large economies.

The results are of interest for several reasons. First, they identify the inherent coordination condition between the network structure and signal structure which allows traders to make Bayesian inference based on a sufficient statistic (or a simple average) of observed signals to reduce the potential computational burden of Bayesian inference. Second, they reveal the relation and essential distinction between the two information processing approaches and add to the understanding of the impact of social networks on traders' decision-making and market outcomes under the framework of rational expectations equilibrium (REE). Finally, the results establish that the equivalence of the two economies is determined by the conditional mean without considering the conditional variance which is involved in the expression of traders' optimal demands. This observation may be of wider significance in other CARA-normality circumstances.

Outline of the Paper. The rest of the paper is organized as follows: Section 2 introduces the model and formulates the problem. Section 3 presents the main results on the characterizations of the equivalence of the two finite-agent economies. Section 4 considers large economies. Section 5 presents the related literature. Section 6 concludes this paper. All proofs are presented in the Appendix.

2. The model

Consider the rational expectations equilibrium (Hellwig, 1980) model with an extension to permit general signal structure. This is a single-period model and there is a single risky asset in the economy. The risky asset has fundamental value θ, and is in fixed supply $X \in \mathbb{R}$, which is normalized to zero for simplifying the exposition. Traders cannot directly observe the fundamentals; instead, they receive signals about the fundamentals of the risky asset.

There are finitely many traders in the economy. Each trader has a CARA utility and maximizes the conditional expected utility of their net profit W based on their information set $\mathcal{F}$:

$$\mathbb{E}\left[-\exp\{-\rho W\}|\mathcal{F}\right],$$

where ρ is the trader's CARA coefficient, $W = W_0 + x(\theta - p)$, where W_0 is the trader's initial wealth, x denotes the holdings of the risky asset, and p denotes its price, which is publicly observable. To prevent the price from being fully revealing, we make the standard assumption in the literature that there is per-capita noise demand u in the economy (i.e., the total noise demand equals nu).

Each trader i ($i = 1, \ldots, n$) initially holds a (one-dimensional) private signal y_i. We impose the following assumption throughout the study.

[3] As stated in Ozsoylev and Walden (2011), the averaging method satisfies some reasonable properties: (i) Traders with more neighbors will receive more precise signals about the risky asset; (ii) Two traders who have the same neighbor set receive the same signal; (iii) All else equal, two traders' signals have a higher correlation if they are connected than not connected.

[4] An inexplicit assumption is that traders can observe the complete collection of their neighbors' signals.

Assumption 1. All random variables are normally distributed, with means normalized to zero and positive variance. The noise trade u is independent of all other random variables in the economy. The variance–covariance matrix of random vector $(\theta, \boldsymbol{y}')'$ is positive definite, and $\mathrm{Cov}(\theta, y_i) > 0$ for every i, where $\boldsymbol{y} = (y_1, \ldots, y_n)'$ is the complete collection of all the signals in the economy. Furthermore, $\mathrm{Var}(\theta|\bar{\boldsymbol{y}}) > \mathrm{Var}(\theta|\boldsymbol{y})$ for any $\bar{\boldsymbol{y}}$ which is a strict subset of $\boldsymbol{y}$.

The normality assumption is standard in the literature. The non-degeneracy assumption on $(\theta, \boldsymbol{y}')'$ is weak and requires that any trader's (initial) signal cannot be pinned down by other signals in the economy, and that the fundamental can also not be pinned down by the complete collection of all signals in the economy. The inequality on conditional variance means that no signal is redundant to predict the fundamental and the complete collection of all signals in the economy is strictly informationally superior to any of its strict subsets. This is reasonable and satisfied for the classical signal structure of a sum of the fundamental and an independent noise (see Assumption 3).

Following the literature, in this study we consider linear equilibria and focus on the class of linear price functions which are linear functions of signals and noise trade. Under the CARA-normality setting, it is well-known that the optimal demand for the risky asset by trader i is given by

$$x_i^* = \frac{\mathbb{E}(\theta|\mathcal{F}_i) - p}{\rho_i \,\mathrm{Var}(\theta|\mathcal{F}_i)}, \tag{1}$$

where ρ_i is trader i's risk aversion coefficient, and $\mathcal{F}_i$ is trader i's information set, which consists of normal random variables and will be described in detail below. As the price is publicly observable, $p \in \mathcal{F}_i$ for every i. A rational expectations equilibrium (REE in short) consists of equilibrium price p and optimal demand x_i^* given by (1), such that the following market-clearing condition holds:

$$\sum_{i=1}^{n} x_i^* + nu = \sum_{i=1}^{n} \frac{\mathbb{E}(\theta|\mathcal{F}_i) - p}{\rho_i \,\mathrm{Var}(\theta|\mathcal{F}_i)} + nu = 0.$$

We now consider an environment of social networks. Investors trade in an exogenously given social network, which is *strongly connected* and described by a directed graph $\mathcal{G}$.[5] Besides their own private signal y_i, each trader i can also observe their neighbors' signals $\{y_j, j \in \mathcal{N}_i\}$ via the social network[6]; here $\mathcal{N}_i$ denotes the neighbor set of trader i in the network. As traders know their own signals, we naturally assume that $i \in \mathcal{N}_i$ for every i. For notational convenience, we denote $\boldsymbol{y}_i = (y_j, j \in \mathcal{N}_i)$ as the observed signals by trade i. There are two potential approaches for traders to process observed signals:

- **Approach 1**: Traders directly infer information about the fundamental of the risky asset from the complete collection $\boldsymbol{y}_i$ of observed signals. In this case, the information set of trader i is given by $\{\boldsymbol{y}_i, p\} =: \mathcal{F}_i^1$.
- **Approach 2**: Traders indirectly infer information about the fundamental from the conditional mean $\mathbb{E}(\theta|\boldsymbol{y}_i)$. In this case, the information set of trader i is given by $\{\mathbb{E}(\theta|\boldsymbol{y}_i), p\} =: \mathcal{F}_i^2$.

Two remarks are in order here. First, the model setting is the same as in the Hellwig (1980) model except for the two differences listed below. The first one is that the signal structure in our model is general. Besides the normality and non-degeneracy requirements made in Assumption 1, we impose no more requirements. The second one is that, besides their own private signal, each trader can also obtain their neighbors' signals. Second, we illustrate the reason traders infer information from a conditional mean instead of an averaged signal, as described in **Approach 2**. Ozsoylev and Walden (2011) consider a special signal structure $y_i = \theta + \epsilon_i$, where the noise terms $\{\epsilon_i\}$ are normally distributed, and jointly independent across traders with mean zero and identical variance. In the benchmark model of Ozsoylev and Walden (2011), each trader takes an average of her neighbors' signals to make Bayesian inference. By the projection theorem for normal random variables (Lemma 1 in the Appendix), the conditional mean $\mathbb{E}(\theta|\boldsymbol{y}_i)$ equals the average $\frac{1}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i} y_j$ of signals up to multiplying it by a constant,[7] that is, the conditional mean $\mathbb{E}(\theta|\boldsymbol{y}_i)$ is informationally equivalent to the average $\frac{1}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i} y_j$. However, for general signal structure or the special signal structure discussed above but with different precisions, the conditional mean of a group of signals is no longer informationally equivalent to their average. As the signal structure in our model is general, it is reasonable for us to consider a conditional mean instead of an average of signals.

For the two information processing approaches (i.e., **Approach 1** and **Approach 2**), we obtain two corresponding (rational expectations) economies, denoted as **Economy 1** and **Economy 2**, respectively.[8] The two economies are similar, except that traders process their neighbors' information differently. The two economies can be nested in the economy in Lou ⓡ al. (2019) which permits signals to be multidimensional and to present arbitrary correlation pattern. Proposition 3 in Lou ⓡ al. (2019) tells us that for each of the two economies, there exists a linear REE and every linear REE is regular. That is, there exist $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)' \neq \boldsymbol{0}$ and $\gamma > 0$ such that $p = \boldsymbol{\pi}'\boldsymbol{y} + \gamma u$ is an equilibrium price.[9] We assume that the linear equilibrium in the two economies is unique,[10] and let p_1 and p_2 denote the respective (unique) linear, regular equilibrium price, and $x_{i,1}^*$ and $x_{i,2}^*$ the respective optimal demand for the risky asset by trader i.

5 A social network is said to be strongly connected if for any two nodes in the network, there exists a path from one node to the other one.

6 Ozsoylev and Walden (2011) and Walden (2019) assume that traders truthfully reveal their private information to their neighbors. This assumption is reasonable in large economies because any trader has no impact on prices even though the trader has informational advantage. But in small economies, the incentive to lie about their private information to their neighbors indeed exists. There possibly exist punishment mechanisms that can guarantee that traders voluntarily reveal their information; see Footnote 14 in Ozsoylev and Walden (2011) and the arguments in Section 3.1 in Walden (2019) for more explanations. A full analysis on such punishment mechanisms is beyond the scope of this paper. In this paper, we mainly characterize the equivalence of the two different information processing approaches, as shown below, based on the presupposition of truthful information sharing between agents.

7 Note that all the random variables are normalized to have mean zero in Assumption 1.

8 **Economies 1** and **2** arise in some specific settings. **Economy 1** arises when some trader is willing to share or sell her information to the other traders for their join benefit (also refer to Indjejikian et al., 2014 for a strategical consideration). Meanwhile, **Economy 2** arises when some trader can observe the order flow of the other traders (Yang and Zhu, 2020). In this case, the trader will also infer the other traders' private information hidden in the order flow besides her own private information and the price. As the order flow is a linear function of the average of the other traders' signals and the price (under the assumption of the homogeneity of the utility function and the signal precisions of traders), the order flow and the price together are informationally equivalent to the information set of the averaged signal and the price. Consequently, the resulting economy can be equivalently reformulated in terms of **Economy 2**.

9 In **Economy 2**, we can view $\mathbb{E}(\theta|\boldsymbol{y}_i)$ as the one-dimensional "signal" of trader i. Then by Proposition 3 in Lou ⓡ al. (2019), we know that there exist vector $(q_1, \ldots, q_n)'$ and $\gamma > 0$ such that $\sum_{i=1}^{n} q_i \mathbb{E}(\theta|\boldsymbol{y}_i) + \gamma u$ is the equilibrium price. Observing that $\mathbb{E}(\theta|\boldsymbol{y}_i)$ is a linear function of signals $\boldsymbol{y}_i$, the equilibrium price $\sum_{i=1}^{n} q_i \mathbb{E}(\theta|\boldsymbol{y}_i) + \gamma u$ can be rewritten as a form of $\boldsymbol{\pi}'\boldsymbol{y} + \gamma u$. Note that the weight vector $\boldsymbol{\pi}$ and γ may differ for the two economies. In addition, because all random variables are normalized to have mean zero, there is no intercept term in the price function.

10 The linear equilibrium is indeed unique when traders are sufficiently risk averse, see Proposition 4 in Lou ⓡ al. (2019).

We argue that the two economies are *equivalent* if $p_1 = p_2$ and $x^*_{i,1} = x^*_{i,2}$, $i = 1, \ldots, n$, almost surely.

It is important to investigate the equivalence of the two economies. When the two economies are equivalent, it is reasonable that traders use the conditional mean, instead of the complete collection, of observed signals to make Bayesian inference. When the two economies are not equivalent, **Economy 2** is not *"stable"* in the sense that there exists at least one trader in this economy who has an incentive to use the complete collection of observed signals from their neighbors to increase their expected utility, besides the conditional mean, as indicated by the following proposition.

Proposition 1. *When the two economies are not equivalent, in* ***Economy 2*** *there exists at least one trader i whose expected utility at the demand* $\frac{\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i),\boldsymbol{y}_i,p_2)-p_2}{\rho_i \operatorname{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i),\boldsymbol{y}_i,p_2)}$[11] *is strictly greater than that at* $\frac{\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i),p_2)-p_2}{\rho_i \operatorname{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i),p_2)}$.

We can also similarly show that there still exists at least one trader who has an incentive to use the complete collection of their neighbor's signals to increase their expected utility when the two resulting economies are still not equivalent (where the two resulting economies are the same as before except that in **Economy 2**, traders also use the collection of their neighbors' information when they have such an incentive as shown in Proposition 1). The phenomenon occurs repeatedly until all traders use the complete collection of their neighbors' information to make Bayesian inference, as happens in **Economy 1**. The above arguments show that when the two economies are not equivalent, there is no reason why traders use an averaged signal rather than the complete collection of observed signals to make Bayesian inference and consequently validate the claim in Lou ⓡ al. (2019).

The rest of this paper will be devoted to investigating *whether and under what conditions* the two economies are equivalent.

Notations. The operator Var stands for the variance of a random variable; and boldface **Var**, for the variance–covariance matrix of a random vector. Similarly, Cov stands for covariance; and boldface **Cov**, for a vector of covariances. For instance, $\operatorname{Var}(p)$ will be the scalar variance of price p, whereas $\mathbf{Var}(\boldsymbol{y})$ stands for the variance–covariance matrix of random vector $\boldsymbol{y}$. For any random variable x and random vector $\boldsymbol{z} = (z_1, \ldots, z_t)'$, $\mathbf{Cov}(x, \boldsymbol{z})$ is shorthand for the vector of covariances $(\operatorname{Cov}(x, z_1), \ldots, \operatorname{Cov}(x, z_t))'$, where $'$ denotes the transpose of a vector.

3. Equivalence characterization

In this section, we first present necessary and sufficient conditions for the equivalence of the two economies, and then establish two impossibility results which reveal that the two economies are not equivalent in general. We finally analyze the equivalence for a classical signal structure commonly used in the literature. In what follows we also refer to p as the equilibrium price of one of the two economies when we do not explicitly specify which one.

3.1. Necessary and sufficient conditions

The following proposition presents a necessary and sufficient condition for the equivalence of the two economies in terms of conditional mean.

Proposition 2. *The two economies are equivalent if and only if one equilibrium price p of the two economies satisfies that* $\mathbb{E}(\theta|\boldsymbol{y}_i, p) = \mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p)$ *almost surely for every i.*

Proposition 2 simplifies the equivalence condition in the definition. To verify the equivalence of the two economies by the definition, it needs to be checked whether, besides the equilibrium price, the optimal demands of traders, which involve conditional mean and variance, in the two economies are respectively equal. However, Proposition 2 shows that the equivalence of the two economies is determined by the conditional mean. The following result shows that the two economies are equivalent (for any signal structure) if the network graph is complete, that is, $\mathcal{N}_i = \{1, \ldots, n\}$ for every i.

Theorem 1. *The two economies are equivalent if the network graph is complete.*

When the network graph is complete, every trader can access all the information in the economy. The full information is a sufficient statistic in the estimation of the fundamental and the price is therefore redundant because it cannot help traders achieve higher utility. In addition, it is clear that the conditional mean of signals is also a sufficient statistic for the complete collection of all signals in the economy using Bayesian updating. The two economies are thus equivalent for complete graphs.

We continue to consider the equivalence for non-complete graphs. Next, we impose an assumption, which requires that, besides the "signal" $\mathbb{E}(\theta|\boldsymbol{y}_i)$, price p can offer more information to predict the fundamental. We intuitively show, in Remark 2, that it indeed holds for the classical signal structure commonly-used in the literature.

Assumption 2. The equilibrium price p in **Economy 2** satisfies that
$\operatorname{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p) < \operatorname{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i))$ for any i with $\mathcal{N}_i \neq \{1, \ldots, n\}$.

The following result characterizes the functional structure of the candidate equilibrium price under which the two economies are equivalent.

Proposition 3. *Suppose the network graph is not complete and Assumption 2 holds. Then, a necessary condition for the equivalence of the two economies is that the two economies have the same equilibrium price p and the equilibrium price takes the form* $p = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u$, *where* $c > 0$ *and* $\gamma > 0$ *are constants.*

Remark 1. The price function $p = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u$ indeed satisfies Assumption 2 by noting Lemma 4 in the Appendix and the three relations that $\operatorname{Cov}(\theta, p) = c\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}))$ (Lemma 3 (i)),

$$\operatorname{Cov}(\mathbb{E}(\theta|\boldsymbol{y}_i), p) = c\operatorname{Cov}(\mathbb{E}(\theta|\boldsymbol{y}_i), \mathbb{E}(\theta|\boldsymbol{y})) = c\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))$$

(Lemma 3 (ii)) and $\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y})) > \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))$ for any i with $\mathcal{N}_i \neq \{1, \ldots, n\}$, where the inequality is due to the relation $\operatorname{Var}(\theta|\boldsymbol{y}) < \operatorname{Var}(\theta|\boldsymbol{y}_i)$ by Assumption 1, and the two equalities $\operatorname{Var}(\theta|\boldsymbol{y}) = \operatorname{Var}(\theta) - \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}))$ and $\operatorname{Var}(\theta|\boldsymbol{y}_i) = \operatorname{Var}(\theta) - \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))$ (the Law of Total Variance).

Proposition 3 establishes a necessary condition for the equivalence of the two economies. It shows that it is necessary for the equivalence that the signal part in the price function perfectly aggregates all the information in the economy. We remark that although signals enter the price function $p = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u$ in a perfect Bayesian updating way, the price function $p = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u$ is not an equilibrium price in general.[12] It is also worth noting

[11] It is clear that $\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), \boldsymbol{y}_i, p_2) = \mathbb{E}(\theta|\boldsymbol{y}_i, p_2)$ and thus,
$\operatorname{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), \boldsymbol{y}_i, p_2) = \operatorname{Var}(\theta|\boldsymbol{y}_i, p_2)$
by the Law of Total Variance.

[12] As shown by Theorem 2 in this paper, the price function $p = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u$ is an equilibrium price in **Economy 2** if and only if the system of equations (5) and (6) has a positive solution (c, γ). However, the system of equations (5) and (6) generally has no positive solution unless under special circumstances.

that the signal part of a price function cannot perfectly aggregate information in the presence of noise trade in general,[13] although it can when there is no noise trade or the noise trade is small enough; see Grossman (1976), Hellwig (1980) and Lou ⓡ al. (2019).

We next discuss the sufficiency condition. Precisely, we investigate the two questions below:

(i) Does the price function $p = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u$ almost surely satisfy the relation $\mathbb{E}(\theta|\boldsymbol{y}_i, p) = \mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p)$ for every i?

(ii) Do constants $c > 0$ and $\gamma > 0$ exist such that $p = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u$ is an equilibrium price? More specifically, such that the price function $p = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u$ satisfies the following market-clearing condition?

$$\sum_{i=1}^{n} \frac{\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p) - p}{\rho_i \operatorname{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p)} + nu = 0. \tag{2}$$

The following proposition affirmatively answers the first question (i).

Proposition 4. *For any $c > 0$ and $\gamma > 0$, $p = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u$ almost surely satisfies that $\mathbb{E}(\theta|\boldsymbol{y}_i, p) = \mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p)$ for every i.*

Proposition 4 shows that under the price function $p = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u$,[14] the two information processing approaches lead to the same optimal demand by any trader. In general, the information set $\{\boldsymbol{y}_i, p\}$ offers more information than $\{\mathbb{E}(\theta|\boldsymbol{y}_i), p\}$. However, when the signal part of the price function perfectly aggregates information, the two information sets are equally informative.

We now address the second question (ii) of whether the price function $p = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u$ is an equilibrium price for appropriately-selected parameters c and γ. Denote

$$a_i := \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y})) - \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) \geq 0.$$

By equality (A.3) in the Appendix with the identity $\boldsymbol{s} = \boldsymbol{y}_i$ and $z = p$, along with (A.12)–(A.14) in the Appendix, we see that for the price function $p = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u$, the conditional mean is given by

$$\begin{aligned}\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p) &= \frac{\gamma^2 \operatorname{Var}(u)}{\gamma^2 \operatorname{Var}(u) + c^2 a_i}\mathbb{E}(\theta|\boldsymbol{y}_i) + \frac{c a_i}{\gamma^2 \operatorname{Var}(u) + c^2 a_i} p \\ &=: \alpha_i \mathbb{E}(\theta|\boldsymbol{y}_i) + \beta_i p,\end{aligned} \tag{3}$$

and the conditional variance is thus given by

$$\begin{aligned}\operatorname{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p) &= \operatorname{Var}(\theta) - [\alpha_i \operatorname{Cov}(\theta, \mathbb{E}(\theta|\boldsymbol{y}_i)) + \beta_i \operatorname{Cov}(\theta, p)] \\ &= \operatorname{Var}(\theta) - [\alpha_i \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) + \beta_i c \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}))] \\ &= \operatorname{Var}(\theta) - \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) - \frac{c^2 a_i^2}{\gamma^2 \operatorname{Var}(u) + c^2 a_i},\end{aligned} \tag{4}$$

where the second equality follows from (A.12) and (A.14) in the Appendix, and the third equality from the definitions of α_i, β_i and a_i. Substituting $p = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u$ into the market-clearing condition (2), and matching the terms, we get the following system of equations:

$$\gamma \sum_{i=1}^{n} \frac{\gamma^2 \operatorname{Var}(u) + (c^2 - c) a_i}{n\rho_i[\gamma^2 \operatorname{Var}(u) + c^2 a_i]\big[\operatorname{Var}(\theta) - \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) - \frac{c^2 a_i^2}{\gamma^2 \operatorname{Var}(u) + c^2 a_i}\big]} = 1,$$

$$\begin{aligned}&c \sum_{i=1}^{n} \frac{\gamma^2 \operatorname{Var}(u) + (c^2 - c) a_i}{\rho_i[\gamma^2 \operatorname{Var}(u) + c^2 a_i]\big[\operatorname{Var}(\theta) - \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) - \frac{c^2 a_i^2}{\gamma^2 \operatorname{Var}(u) + c^2 a_i}\big]}\mathbb{E}(\theta|\boldsymbol{y}) \\ &= \sum_{i=1}^{n} \frac{\gamma^2 \operatorname{Var}(u)}{\rho_i[\gamma^2 \operatorname{Var}(u) + c^2 a_i]\big[\operatorname{Var}(\theta) - \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) - \frac{c^2 a_i^2}{\gamma^2 \operatorname{Var}(u) + c^2 a_i}\big]}\mathbb{E}(\theta|\boldsymbol{y}_i)\end{aligned}$$

almost surely,

or, equivalently, in a simplified form

$$\gamma \sum_{i=1}^{n} \frac{\gamma^2 \operatorname{Var}(u) + (c^2 - c) a_i}{n\rho_i\{\gamma^2 \operatorname{Var}(u) a_i + [\gamma^2 \operatorname{Var}(u) + c^2 a_i][\operatorname{Var}(\theta) - \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}))]\}} = 1, \tag{5}$$

$$\begin{aligned}&\gamma \sum_{i=1}^{n} \frac{\gamma^2 \operatorname{Var}(u)}{n\rho_i\{\gamma^2 \operatorname{Var}(u) a_i + [\gamma^2 \operatorname{Var}(u) + c^2 a_i][\operatorname{Var}(\theta) - \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}))]\}}\mathbb{E}(\theta|\boldsymbol{y}_i) \\ &= c\mathbb{E}(\theta|\boldsymbol{y}) \text{ almost surely}\end{aligned} \tag{6}$$

From the above analysis, together with Propositions 2–4, we get a necessary and sufficient condition for the equivalence of the two economies.[15]

Theorem 2. *Suppose Assumption 2 holds. Then, the two economies are equivalent if and only if the system of equations (5) and (6) has a positive solution (c, γ).*[16]

3.2. Impossibility results

The following two theorems reveal that the two economies are generally not equivalent.

Theorem 3. *For any signal structure that satisfies Assumption 1, there always exists a non-complete network graph, which depends on the given signal structure, such that the two economies are not equivalent under this graph.*

Theorem 4. *For any non-complete network graph, there always exists a signal structure, which depends on the given network graph and satisfies Assumption 1, such that the two economies are not equivalent under this signal structure.*

In the proof, for any given signal structure (respectively, non-complete graph), we can construct a corresponding network graph (respectively, signal structure) such that the system of equations (5) and (6) has no positive solution. Theorems 4 and 2 reveal that the two economies are equivalent for any signal structure if and only if the network graph is a complete graph.

13 That is, although Proposition 3 in Lou ⓡ al. (2019) tells us that there exist $(c_1, \ldots, c_n)'$ and $\gamma > 0$ such that the equilibrium price in **Economy 2** is of the form $p = \sum_{i=1}^{n} c_i \mathbb{E}(\theta|\boldsymbol{y}_i) + \gamma u$, $\sum_{i=1}^{n} c_i \mathbb{E}(\theta|\boldsymbol{y}_i)$, which cannot be described as the form of $c\mathbb{E}(\theta|\boldsymbol{y})$ for some $c > 0$ in general.

14 Note again that price function $p = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u$ is not an equilibrium price in general, see Footnote 12 for more explanation.

15 When the network graph is complete, $\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) = \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}))$ and consequently, $a_i = 0$ for every i. Hence the system of equations (5) and (6) has a positive solution:

$$c = 1 \text{ and } \gamma = \frac{\operatorname{Var}(\theta|\boldsymbol{y})}{\sum_{i=1}^{n} \frac{1}{\rho_i}}.$$

This result is consistent with Theorem 1.

16 We remark that when the network graph is not complete, if the system of equations (5) and (6) has a positive solution (c, γ), then it must hold that $c < 1$. To see this, first observe that when the network graph is not complete, $a_i > 0$ for at least one i by the equality (4) and Assumption 2. Then taking covariance $\operatorname{Cov}(\theta, \cdot)$ of both sides of (6) and arranging the term gives

$$\gamma \sum_{i=1}^{n} \frac{\gamma^2 \operatorname{Var}(u)}{\rho_i\{\gamma^2 \operatorname{Var}(u) a_i + [\gamma^2 \operatorname{Var}(u) + c^2 a_i][\operatorname{Var}(\theta) - \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}))]\}} \times \frac{\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))}{\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}))} = c,$$

where we use Lemma 3 (i). Finally, using an argument by contradiction, it follows from the preceding equality, the relation $\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y})) \geq \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))$, that is, $a_i \geq 0$ for every i (the strict inequality holds for at least one i) and the equality (5) that $c < 1$.

An important observation from the preceding two impossibility results, as well as Theorem 2, is that for the equivalence of the two economies, the network structure and signal structure need to coordinate well.

3.3. The classical signal structure

For the general signal structure, it is extremely difficult to identify whether the system of equations (5) and (6) has a positive solution. To get a more precise characterization, in this subsection we consider the classical signal structure commonly used in the literature (see, e.g., Grossman, 1976, Hellwig, 1980, Ozsoylev and Walden, 2011 and Han and Yang, 2013).

Assumption 3. The signal structure takes the form of $y_i = \theta + \epsilon_i$, where the noise terms $\{\epsilon_i\}$ are normally distributed with mean zero, independent of each other and of all other random variables in the economy.

Remark 2. Here, we test Assumption 2 under Assumption 3. Let $p = \boldsymbol{\pi}'\boldsymbol{y} + \gamma u$ be the equilibrium price in **Economy 2**. Under Assumption 3, letting $\tau_\theta = 1/\mathrm{Var}(\theta)$ and $\tau_{\epsilon_j} = 1/\mathrm{Var}(\epsilon_j)$, by the projection theorem for normal random variables (Lemma 1 in the Appendix; also refer to page 378 in Vives, 2008), we obtain

$$\mathbb{E}(\theta|\boldsymbol{y}_i) = \frac{\sum_{j\in\mathcal{N}_i}\tau_{\epsilon_j}y_j}{\tau_\theta + \sum_{j\in\mathcal{N}_i}\tau_{\epsilon_j}} = \frac{\sum_{j\in\mathcal{N}_i}\tau_{\epsilon_j}}{\tau_\theta + \sum_{j\in\mathcal{N}_i}\tau_{\epsilon_j}}\theta + \frac{\sum_{j\in\mathcal{N}_i}\tau_{\epsilon_j}\epsilon_j}{\tau_\theta + \sum_{j\in\mathcal{N}_i}\tau_{\epsilon_j}}.$$

Consequently,

$$\begin{aligned}&\mathrm{Cov}(\theta - \mathbb{E}(\theta|\boldsymbol{y}_i), p)\\ &= \mathrm{Cov}\Big(\frac{\tau_\theta}{\tau_\theta + \sum_{j\in\mathcal{N}_i}\tau_{\epsilon_j}}\theta - \frac{\sum_{j\in\mathcal{N}_i}\tau_{\epsilon_j}\epsilon_j}{\tau_\theta + \sum_{j\in\mathcal{N}_i}\tau_{\epsilon_j}}, \sum_{i=1}^n \pi_i(\theta+\epsilon_i)\Big)\\ &= \frac{1}{\tau_\theta + \sum_{j\in\mathcal{N}_i}\tau_{\epsilon_j}}\Big(\sum_{i=1}^n \pi_i - \sum_{j\in\mathcal{N}_i}\pi_j\Big).\end{aligned}$$

Hence, $\mathrm{Cov}(\theta - \mathbb{E}(\theta|\boldsymbol{y}_i), p) \neq 0$ for any i with $\mathcal{N}_i \neq \{1,\ldots,n\}$ and then Assumption 2 holds under Assumption 3 by Lemma 4 in the Appendix if the weight assigned to each signal y_i in the equilibrium price $p = \boldsymbol{\pi}'\boldsymbol{y} + \gamma u$ is positive, that is, $\pi_i > 0$ for every i. Proposition 7 (iv) in Lou ⓡ al. (2019) indicates that when the variance of noise trade is sufficiently large, every Q_i, which is the (scaled) weight given to the "signal" $\mathbb{E}(\theta|\boldsymbol{y}_i)$, is positive. From the equality $p = \gamma\sum_{i=1}^n Q_i\mathbb{E}(\theta|\boldsymbol{y}_i) + \gamma u$ and the expression for $\mathbb{E}(\theta|\boldsymbol{y}_i)$ given above, we conclude that when the noise trade is sufficiently large, every π_i is indeed positive.

Under the signal structure in Assumption 3, we now present a more explicit necessary and sufficient condition for the equivalence.

Corollary 1. *Suppose Assumptions 2 and 3 hold, the noise terms $\{\epsilon_i\}$ have the same variance, and traders have homogeneous preferences in the sense that they have the same risk aversion coefficient ρ. The two economies are then equivalent if and only if the following system of $(n+1)$ equations has a positive solution (c, γ):*

$$\frac{\gamma}{n\rho}\sum_{i=1}^n \frac{\gamma^2\,\mathrm{Var}(u) + (c^2-c)\big(\frac{1}{\tau_\theta+|\mathcal{N}_i|\tau_\epsilon} - \frac{1}{\tau_\theta+n\tau_\epsilon}\big)}{\gamma^2\,\mathrm{Var}(u)\frac{1}{\tau_\theta+|\mathcal{N}_i|\tau_\epsilon} + c^2\big(\frac{1}{\tau_\theta+|\mathcal{N}_i|\tau_\epsilon} - \frac{1}{\tau_\theta+n\tau_\epsilon}\big)\frac{1}{\tau_\theta+n\tau_\epsilon}} = 1, \tag{7}$$

$$\frac{\gamma^3\,\mathrm{Var}(u)}{cn\rho}\sum_{\{j|i\in\mathcal{N}_j\}}\frac{\tau_\theta+n\tau_\epsilon}{\gamma^2\,\mathrm{Var}(u) + c^2\frac{(n-|\mathcal{N}_j|)\tau_\epsilon}{(\tau_\theta+n\tau_\epsilon)^2}} = 1,\ \ i = 1,\ldots,n. \tag{8}$$

Corollary 2. *Suppose the network graph is undirected,*[17] *and let the conditions in Corollary 1 hold. Then*

- *the two economies are equivalent if the network graph is regular, and*
- *the two economies are not equivalent if the network graph is a chain or a star.*[18]

4. A large economy

In this section, we consider a large economy which can justify the assumption that traders are price-takers and are willing to share information with their neighbors. Here, we assume that traders' signals take the form in Assumption 3 and consider the same network setting as the replica economy in Han and Yang (2013) and Walden (2019).

We first define an m-replica economy $\mathcal{A}_m$ as one which consists of m disjoint identical replicas of the network introduced in Section 2. The m replicas have equal network size (i.e., n) and identical network structure. In the economy $\mathcal{A}_m$, noise terms in signals are independent across traders. The total noise trade in the economy $\mathcal{A}_m$ equals nmu, where nm is the size of economy $\mathcal{A}_m$, and u is a normal random variable with mean zero and independent of all other random variables in the economy. We then consider a sequence of replica economies $\{\mathcal{A}_m\}_{m\geq 1}$ and define the large economy as the limit $\mathcal{A} := \lim_{m\to\infty}\mathcal{A}_m$; see Walden (2019) for more illustrations on replica economies. We also define two large economies, denoted $\mathcal{A}^1$ and $\mathcal{A}^2$, corresponding to **Approach 1** and **Approach 2**, analogous to those in Section 2. Like the finite-agent economies in Section 2, the two large economies are identical except for differing information processing approaches.

The following theorem shows that the two large economies are equivalent.

Theorem 5. *Let Assumption 3 hold and traders have homogeneous preferences in the sense that they have the same risk aversion coefficient ρ. For any network structure, the two large economies $\mathcal{A}^1$ and $\mathcal{A}^2$ are equivalent. The identical equilibrium price in the two large economies is given by*

$$p = \frac{1}{\sum_{i=1}^n\Big(\tau_\theta + \sum_{j\in\mathcal{N}_i}\tau_{\epsilon_j}\Big)}\left(\sum_{i=1}^n\sum_{j\in\mathcal{N}_i}\tau_{\epsilon_j}\theta + n\rho u\right),$$

where $\tau_\theta = \frac{1}{\mathrm{Var}(\theta)}$ and $\tau_{\epsilon_j} = \frac{1}{\mathrm{Var}(\epsilon_j)}$.

Although the two information processing approaches in finite-agent economies are generally not equivalent, as shown in Section 4, they are indeed equivalent in large economies (for the classical signal structure), as shown by Theorem 5. This justifies the assumption that traders take an average of their neighbors' signals to make Bayesian inference in Ozsoylev and Walden (2011). Intuitively, when the economies are large, the noise terms do not enter into prices such that the equilibrium prices take a linear form of the fundamental and an independent term (i.e., the noise demand u). Under such a price structure, the projection theorem for normal random variables tells us that the conditional estimates based on a group of signals with a fundamental-noise structure are the same regardless of whether some signals are replaced by their conditional means.

[17] A graph is said to be undirected if the arc (i,j) belongs to this graph, then the arc (j,i) also belongs to this graph.

[18] An undirected graph is said to be k-regular if each node has exactly k neighbors, a chain if the arc set is given by $\{(i,i+1), i = 1,\ldots,n-1\}$ up to a permutation in node index, and a star if the arc set is given by $\{(1,i), i = 2,\ldots,n\}$ up to a permutation in node index.

5. Related literature

Our study is related to the vast literature on REE with a Gaussian signal structure.[19] Our work investigates the equivalence of the two economies with different information processing approaches built upon the Hellwig (1980) model with an extension to allow for general signal structure. The resulting two noisy REE models can be nested in the model in Lou ⓡ al. (2019). However, different from Lou ⓡ al. (2019), which shows the existence and regularity of linear equilibrium, and establishes an information aggregation result in the presence of small noise trade, our work focuses on the equivalence of the two resulting economies. Noteworthily, the equilibrium existence and regularity results in Lou ⓡ al. (2019) can be directly applied to establish the existence and regularity of a linear equilibrium of the two economies considered here.

Our work is also related to the recent strand of theoretical literature on the implications of information sharing/diffusion/disclosure on market outcomes,[20] such as efficiency, liquidity, volatility, volume, and welfare, in the framework of REE (see, e.g., Colla and Mele, 2010, Han and Yang, 2013, Manela, 2014, Indjejikian et al., 2014, Goldstein and Yang, 2017 and Yang and Zhu, 2020). Compared to the setting when there is no social communication, Colla and Mele (2010) find that a market with a cyclical network structure will be accompanied by higher volume, liquidity, and efficiency. Han and Yang (2013) consider a model in which one trader receives a signal about the fundamental and the other traders receive a noisy version of this signal via social communication. Traders set their demand based on prices and the observed signals from their neighbors. How networks affect market outcomes has been shown to depend crucially on whether the fraction of informed traders is exogenously fixed or endogenously determined at a cost. Manela (2014) analyzes how the speed of information diffusion in a social network affects traders' welfare. Indjejikian et al. (2014) reveal that an insider has an incentive to intentionally leak some of their private information to an unrelated party.

Our study is most closely related to the two works of Ozsoylev and Walden (2011) and Walden (2019). In the finite-agent model in Ozsoylev and Walden (2011)[21] and the dynamic REE model in Walden (2019),[22] information is shared/transmitted via a social network. Traders' signals take the classical form of a sum of the fundamental and independent noise. After observing neighbors' signals, traders infer information about the fundamental from an average of the observed signals as well as from prices. Ozsoylev and Walden (2011) and Walden (2019) study, for their respective models with *large* information networks, how the network topology (such as connectedness and centrality) affects factors such as asset prices, volatility, trading volume, and welfare. Motivated by the approach of averaging neighbors' information, in this study, we build upon the Hellwig (1980) model with the aim of investigating the equivalence of the averaging approach and the most popular approach of inferring information from the complete collection of neighbors' signals. Different from the special signal structure in Ozsoylev and Walden (2011), we consider a more general signal structure and use a sufficient statistic for observed signals to substitute for the averaged signal.

Our study is also related to the literature on Bayesian learning (Banerjee, 1992; Bikhchandani et al., 1992) and non-Bayesian learning (DeGroot, 1974; Demarzo et al., 2003; Jadbabaie et al., 2012; Molavi et al., 2018) over social networks. This literature explores the information aggregation implications of Bayesian/Non-Bayesian inference in an environment where individuals can observe the actions of other agents. Our study differs from this literature in that our model is a static (not a dynamic) one, and agents in it can only observe the signals, *not* the actions, of other agents.

6. Concluding remarks

We revisited the Hellwig (1980) model under an environment in which traders can observe their neighbors' signals via an exogenously given social network. There are two potential approaches for traders to process the observed signals from their neighbors: traders infer information about the fundamental directly from the complete collection of observed signals, or indirectly from a sufficient statistic of observed signals (i.e., the average of observed signals when signals take the classical form of a sum of the fundamental and independent noise). We presented a complete characterization of the necessary and sufficient conditions for the equivalence between the two finite-agent economies corresponding to the two information processing approaches. It shows that the two economies are not equivalent in general, unless the network structure and signal structure coordinate well. For the classical signal structure, we showed that the two economies are equivalent for regular graphs, but not for chain graphs and star graphs. In addition, we also showed that when the signals take the classical form, the two large economies, defined as the limit of a sequence of replica finite-agent economies, are equivalent for any network structure.

Although the conditional mean of the fundamental is a sufficient statistic for the complete collection of the observed signals, the results reveal that the two information processing approaches do not, in general, lead to the same market equilibrium under the framework of REE when the price is also considered as endogenous public information. Our analysis reveals the relation and the essential distinction between the two information processing approaches, and contributes to the literature on the impact of social networks on traders' decision-making and market outcomes.

There are several interesting extensions to be tackled in the future. In this study, we did not consider the cost associated with information acquisition. Although the complete collection of signals always weakly dominates the sufficient statistic of signals as shown by Proposition 1, observing the former may be costlier than the latter in practice. Therefore, the first interesting extension would be to consider the information acquisition cost as done in Grossman and Stiglitz (1980), and investigate the trade-off between the benefit and the cost from observing more informative signals. The second interesting extension would be to apply the developed method in this study to analyze the equivalence of the two information processing approaches in other CARA-normality models. The third one is to extend the current analysis to more general multi-asset settings, where the equilibrium existence in a finite-agent, multi-asset setting has been established (Carpio and Guo, 2019).

[19] See, for example, Grossman (1976), Hellwig (1980), Grossman and Stiglitz (1980), Verrecchia (1982), Vives (2008), Ganguli and Yang (2009), Goldstein and Yang (2015), Rahi and Zigrand (2018), He et al. (2019), Lou and Wang (2020) and Lou ⓡ al. (2019), among many others.

[20] There are also empirical/experimental literature that study the effect of social networks on investment decision-making and market outcomes, see, for example, Feng and Seasholes (2004), Hong et al. (2004), Hong et al. (2005), Brown et al. (2008), Pool et al. (2015), Heimer (2016), Ozsoylev et al. (2014) and Halim et al. (2019), among many others.

[21] It is emphasized that the finite-agent model in Ozsoylev and Walden (2011) only serves as a benchmark to facilitate the later analysis on large economies.

[22] Different from the risk-aversion assumption in Ozsoylev and Walden (2011) and this paper, Walden (2019) builds upon a different REE model in which a risk-neutral competitive market maker is present in the market to facilitate the analysis.

Acknowledgments

The authors would like to thank the editor Yusufcan Masatlioglu, an associate editor, an anonymous referee, Moris Strub and Johan Walden for their helpful comments and suggestions.

Appendix

In this appendix, we present all the proofs. We begin with some useful lemmas. The first one is the standard *projection theorem for normal random variables*, which will be frequently used in the proofs; refer to Chapter 5, Section 4 in DeGroot (1970) for more details.

Lemma 1. *For a normal random vector* $(\theta, \boldsymbol{s}')'$ *with mean zero and positive definite variance–covariance matrix*

$$\begin{pmatrix} \mathrm{Var}(\theta) & \mathbf{Cov}(\theta,\boldsymbol{s})' \\ \mathbf{Cov}(\theta,\boldsymbol{s}) & \mathbf{Var}(\boldsymbol{s}) \end{pmatrix},$$

the conditional mean and variance are respectively given by

$$\mathbb{E}(\theta|\boldsymbol{s}) = \mathbf{Cov}(\theta,\boldsymbol{s})'\mathbf{Var}(\boldsymbol{s})^{-1}\boldsymbol{s}$$

and

$$\mathrm{Var}(\theta|\boldsymbol{s}) = \mathrm{Var}(\theta) - \mathbf{Cov}(\theta,\boldsymbol{s})'\mathbf{Var}(\boldsymbol{s})^{-1}\mathbf{Cov}(\theta,\boldsymbol{s}).$$

It follows from the above projection theorem that

$$\mathrm{Var}(\theta) = \mathrm{Var}(\theta|\boldsymbol{s}) + \mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{s})),$$

which is referred to as *the Law of Total Variance* in the literature.

Lemma 2. *Let* $\boldsymbol{s}$ *be a normal random vector. Then*

$$\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{s})) = \mathbb{E}(\theta|\boldsymbol{s}), \quad \mathrm{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{s})) = \mathrm{Var}(\theta|\boldsymbol{s}).$$

Proof. There is no loss of generality in assuming that $\mathbf{Var}(\boldsymbol{s})$ is positive definite because, otherwise, the desired result can be shown using the arguments below by removing the components in $\boldsymbol{s}$ which can be expressed as a linear combination of the other components in $\boldsymbol{s}$. According to Lemma 1,

$$\mathbb{E}(\theta|\boldsymbol{s}) = \mathbf{Cov}(\theta,\boldsymbol{s})'\mathbf{Var}(\boldsymbol{s})^{-1}\boldsymbol{s}$$

and

$$\mathrm{Var}(\theta|\boldsymbol{s}) = \mathrm{Var}(\theta) - \mathbf{Cov}(\theta,\boldsymbol{s})'\mathbf{Var}(\boldsymbol{s})^{-1}\mathbf{Cov}(\theta,\boldsymbol{s}).$$

From the conditional mean $\mathbb{E}(\theta|\boldsymbol{s})$, it is easily verified that

$$\mathrm{Cov}(\theta,\mathbb{E}(\theta|\boldsymbol{s})) = \mathbf{Cov}(\theta,\boldsymbol{s})'\mathbf{Var}(\boldsymbol{s})^{-1}\mathbf{Cov}(\theta,\boldsymbol{s})$$

and

$$\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{s})) = \mathbf{Cov}(\theta,\boldsymbol{s})'\mathbf{Var}(\boldsymbol{s})^{-1}\mathbf{Cov}(\theta,\boldsymbol{s}).$$

Using Lemma 1 and the two preceding equalities, we obtain

$$\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{s})) = \frac{\mathrm{Cov}(\theta,\mathbb{E}(\theta|\boldsymbol{s}))}{\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{s}))}\mathbb{E}(\theta|\boldsymbol{s}) = \mathbb{E}(\theta|\boldsymbol{s})$$

and

$$\begin{aligned}\mathrm{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{s})) &= \mathrm{Var}(\theta) - \frac{\mathrm{Cov}(\theta,\mathbb{E}(\theta|\boldsymbol{s}))^2}{\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{s}))} \\ &= \mathrm{Var}(\theta) - \mathbf{Cov}(\theta,\boldsymbol{s})'\mathbf{Var}(\boldsymbol{s})^{-1}\mathbf{Cov}(\theta,\boldsymbol{s}) \\ &= \mathrm{Var}(\theta|\boldsymbol{s}).\end{aligned}$$

The conclusion follows. □

Lemma 3. *Let* $\boldsymbol{s}$ *be a normal random vector and* $\bar{\boldsymbol{s}}$ *a subset of* $\boldsymbol{s}$. *Then, the following equalities hold*

(i) $\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{s})) = \mathrm{Cov}(\theta,\mathbb{E}(\theta|\boldsymbol{s}))$;
(ii) $\mathrm{Cov}(\mathbb{E}(\theta|\bar{\boldsymbol{s}}),\mathbb{E}(\theta|\boldsymbol{s})) = \mathrm{Var}(\mathbb{E}(\theta|\bar{\boldsymbol{s}}))$;
(iii) $\mathbf{Cov}(\mathbb{E}(\theta|\boldsymbol{s}),\bar{\boldsymbol{s}}) = \mathbf{Cov}(\theta,\bar{\boldsymbol{s}})$.

Proof. (i) was shown in the proof of Lemma 2, and (ii) follows from the following series of equalities:

$$\begin{aligned}\mathrm{Cov}(\mathbb{E}(\theta|\bar{\boldsymbol{s}}),\mathbb{E}(\theta|\boldsymbol{s})) &= \mathbb{E}[\mathbb{E}(\theta|\bar{\boldsymbol{s}})\mathbb{E}(\theta|\boldsymbol{s})] \\ &= \mathbb{E}[\mathbb{E}(\mathbb{E}(\theta|\bar{\boldsymbol{s}})\mathbb{E}(\theta|\boldsymbol{s})|\bar{\boldsymbol{s}})] \\ &= \mathbb{E}[\mathbb{E}(\theta|\bar{\boldsymbol{s}})\mathbb{E}(\mathbb{E}(\theta|\boldsymbol{s})|\bar{\boldsymbol{s}})] \\ &= \mathbb{E}[\mathbb{E}(\theta|\bar{\boldsymbol{s}})^2] \\ &= \mathrm{Var}(\mathbb{E}(\theta|\bar{\boldsymbol{s}})),\end{aligned}$$

where the first and last equalities follow from the definitions of covariance and variance, and the assumption that the mean of θ equals zero, the second from the definition of conditional expectation, the third from the property *"pulling out what's known"* of conditional expectation, and the fourth from the *tower* property of conditional expectation. By $\mathbb{E}(\theta|\boldsymbol{s}) = \mathbf{Cov}(\theta,\boldsymbol{s})'\mathbf{Var}(\boldsymbol{s})^{-1}\boldsymbol{s}$ (Lemma 1), we have $\mathbf{Cov}(\mathbb{E}(\theta|\boldsymbol{s}),\boldsymbol{s}) = \mathbf{Cov}(\theta,\boldsymbol{s})$, from which (iii) follows. The proof is completed. □

Lemma 4. *Let* $\boldsymbol{s}$ *be a normal random vector and* z *a normal random variable. Then* $\mathrm{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{s}),z) = \mathrm{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{s}))$ *if and only if* $\mathrm{Cov}(\theta - \mathbb{E}(\theta|\boldsymbol{s}),z) = 0$.

Proof. The proof follows by some simple computations using Lemma 1 and the relation $\mathrm{Cov}(\theta,\mathbb{E}(\theta|\boldsymbol{s})) = \mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{s}))$ (Lemma 3 (i)). □

Lemma 5. *Let* $(z,\boldsymbol{s}')'$ *be a normal random vector with mean zero, where* $\boldsymbol{s} \in \mathbb{R}^m$, $m \geq 2$, $\mathbf{Var}((\theta,\boldsymbol{s}')')$ *be positive definite,* $\mathrm{Cov}(z,\theta) \neq 0$, *and* $\mathbf{Cov}(\theta,\boldsymbol{s}) \neq c\mathbf{Cov}(z,\boldsymbol{s})$ *for any* $c \neq 0$. *Then* $\mathbb{E}(\theta|\boldsymbol{s},z) = \mathbb{E}(\theta|\boldsymbol{s})$ *almost surely if* $\mathbb{E}(\theta|\boldsymbol{s},z) = \mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{s}),z)$ *almost surely.*

Proof. We assume without loss of generality that $\mathbf{Var}((\boldsymbol{s}',z)')$ is positive definite, otherwise z can be expressed as a linear combination of $\boldsymbol{s}$ (note that $\mathbf{Var}(\boldsymbol{s})$ is positive definite, which follows from the positive definiteness of $\mathbf{Var}((\theta,\boldsymbol{s}')')$); then, the equality $\mathbb{E}(\theta|\boldsymbol{s},z) = \mathbb{E}(\theta|\boldsymbol{s})$ trivially holds.

Suppose $\mathbb{E}(\theta|\boldsymbol{s},z) = \mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{s}),z)$ almost surely. By virtue of Lemma 1, together with some simple calculations, the conditional mean of θ conditional on $\{\boldsymbol{s},z\}$ is given by

$$\mathbb{E}(\theta|\boldsymbol{s},z) = \boldsymbol{\alpha}'\boldsymbol{s} + \beta z,$$

where

$$\boldsymbol{\alpha} = \left[\mathbf{Var}(\boldsymbol{s}) - \frac{\mathbf{Cov}(z,\boldsymbol{s})\mathbf{Cov}(z,\boldsymbol{s})'}{\mathrm{Var}(z)}\right]^{-1}\left[\mathbf{Cov}(\theta,\boldsymbol{s}) - \frac{\mathrm{Cov}(z,\theta)}{\mathrm{Var}(z)}\mathbf{Cov}(z,\boldsymbol{s})\right], \tag{A.1}$$

$$\beta = \frac{\mathrm{Cov}(z,\theta) - \mathbf{Cov}(\theta,\boldsymbol{s})'\mathbf{Var}(\boldsymbol{s})^{-1}\mathbf{Cov}(z,\boldsymbol{s})}{\mathrm{Var}(z) - \mathbf{Cov}(z,\boldsymbol{s})'\mathbf{Var}(\boldsymbol{s})^{-1}\mathbf{Cov}(z,\boldsymbol{s})}. \tag{A.2}$$

Furthermore, because $\{\mathbb{E}(\theta|\boldsymbol{s}),z\}$ is linearly independent, by Lemma 1 we also have Eq. (A.3) which is given in Box I. Observe the relations that $\mathbb{E}(\theta|\boldsymbol{s}) = \mathbf{Cov}(\theta,\boldsymbol{s})'\mathbf{Var}(\boldsymbol{s})^{-1}\boldsymbol{s}$ (Lemma 1) and $\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{s})) = \mathrm{Cov}(\theta,\mathbb{E}(\theta|\boldsymbol{s}))$ (Lemma 3 (i)). Then, by the linear independence of z and $\boldsymbol{s}$, identifying coefficients on $\boldsymbol{s}$ in (A.1) and (A.3) it is immediate that

$$\begin{aligned}&\left[\mathbf{Var}(\boldsymbol{s}) - \frac{\mathbf{Cov}(z,\boldsymbol{s})\mathbf{Cov}(z,\boldsymbol{s})'}{\mathrm{Var}(z)}\right]^{-1}\left[\mathbf{Cov}(\theta,\boldsymbol{s}) - \frac{\mathrm{Cov}(z,\theta)}{\mathrm{Var}(z)}\mathbf{Cov}(z,\boldsymbol{s})\right] \\ &= \frac{\mathrm{Var}(z)\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{s})) - \mathrm{Cov}(\mathbb{E}(\theta|\boldsymbol{s}),z)\,\mathrm{Cov}(\theta,z)}{\mathrm{Var}(z)\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{s})) - \mathrm{Cov}(\mathbb{E}(\theta|\boldsymbol{s}),z)^2}\mathbf{Var}(\boldsymbol{s})^{-1}\mathbf{Cov}(\theta,\boldsymbol{s}).\end{aligned} \tag{A.4}$$

$$\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{s}), z) = \frac{\{\mathrm{Var}(z)\,\mathrm{Cov}(\theta, \mathbb{E}(\theta|\boldsymbol{s})) - \mathrm{Cov}(\mathbb{E}(\theta|\boldsymbol{s}), z)\,\mathrm{Cov}(\theta, z)\}\mathbb{E}(\theta|\boldsymbol{s}) + \{\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{s}))\,\mathrm{Cov}(\theta, z) - \mathrm{Cov}(\theta, \mathbb{E}(\theta|\boldsymbol{s}))\,\mathrm{Cov}(\mathbb{E}(\theta|\boldsymbol{s}), z)\}z}{\mathrm{Var}(z)\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{s})) - \mathrm{Cov}(\mathbb{E}(\theta|\boldsymbol{s}), z)^2}. \quad \text{(A.3)}$$

Box I.

Multiplying $\mathbf{Var}(\boldsymbol{s}) - \frac{\mathbf{Cov}(z,\boldsymbol{s})\mathbf{Cov}(z,\boldsymbol{s})'}{\mathrm{Var}(z)}$ on both sides of (A.4), we get

$$\mathbf{Cov}(\theta, \boldsymbol{s}) - \frac{\mathrm{Cov}(z,\theta)}{\mathrm{Var}(z)}\mathbf{Cov}(z, \boldsymbol{s}) = \frac{\mathrm{Var}(z)\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{s})) - \mathrm{Cov}(\mathbb{E}(\theta|\boldsymbol{s}), z)\,\mathrm{Cov}(\theta, z)}{\mathrm{Var}(z)\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{s})) - \mathrm{Cov}(\mathbb{E}(\theta|\boldsymbol{s}), z)^2} \times \left[\boldsymbol{I}_m - \frac{\mathbf{Cov}(z,\boldsymbol{s})\mathbf{Cov}(z,\boldsymbol{s})'}{\mathrm{Var}(z)}\mathbf{Var}(\boldsymbol{s})^{-1}\right]\mathbf{Cov}(\theta, \boldsymbol{s}), \quad \text{(A.5)}$$

where $\boldsymbol{I}_m$ denotes the identity matrix in $\mathbb{R}^m$. According to the hypothesis $\mathbf{Cov}(\theta, \boldsymbol{s}) \neq c\mathbf{Cov}(z, \boldsymbol{s})$ for any $c \neq 0$, $\mathrm{Cov}(z, \theta) \neq 0$ and the equality (A.5), we immediately conclude that $\mathbf{Cov}(\theta, \boldsymbol{s}) \neq \mathbf{0}$. Consequently, the two coefficients on both sides of (A.5) must be equal, and we therefore have

$$\mathrm{Cov}(z, \theta) = \mathbf{Cov}(z, \boldsymbol{s})'\mathbf{Var}(\boldsymbol{s})^{-1}\mathbf{Cov}(\theta, \boldsymbol{s}) = \mathrm{Cov}(\mathbb{E}(\theta|\boldsymbol{s}), z), \quad \text{(A.6)}$$

where the second equality follows from the equality $\mathbb{E}(\theta|\boldsymbol{s}) = \mathbf{Cov}(\theta, \boldsymbol{s})'\,\mathbf{Var}(\boldsymbol{s})^{-1}\boldsymbol{s}$ (Lemma 1). As a result, $\beta = 0$ by (A.2) and $\boldsymbol{\alpha} = \mathbf{Var}(\boldsymbol{s})^{-1}\mathbf{Cov}(\theta, \boldsymbol{s})$ by (A.1), (A.4) and (A.6). It follows that

$$\mathbb{E}(\theta|\boldsymbol{s}, z) = \mathbf{Cov}(\theta, \boldsymbol{s})'\mathbf{Var}(\boldsymbol{s})^{-1}\boldsymbol{s} = \mathbb{E}(\theta|\boldsymbol{s}).$$

The proof is completed. □

Proof of Proposition 1. Before providing the proof, we first present a formula for calculating the expected utility at the optimal demand $\frac{\mathbb{E}(\theta|\mathcal{F}_i)-p}{\rho_i \mathrm{Var}(\theta|\mathcal{F}_i)}$ (where $p \in \mathcal{F}_i$):

$$\begin{aligned}
&\mathbb{E}\left[-\exp\left\{-\rho_i \frac{\mathbb{E}(\theta|\mathcal{F}_i) - p}{\rho_i \mathrm{Var}(\theta|\mathcal{F}_i)}(\theta - p)\right\}\right] \\
&= -\mathbb{E}\left[\exp\left\{-\frac{\mathbb{E}(\theta - p|\mathcal{F}_i)^2}{2\,\mathrm{Var}(\theta|\mathcal{F}_i)}\right\}\right] \\
&= -\left(1 + \frac{\mathrm{Var}(\theta - p) - \mathrm{Var}(\theta - p|\mathcal{F}_i)}{\mathrm{Var}(\theta|\mathcal{F}_i)}\right)^{-\frac{1}{2}} \\
&= -\sqrt{\frac{\mathrm{Var}(\theta|\mathcal{F}_i)}{\mathrm{Var}(\theta - p)}}, \quad \text{(A.7)}
\end{aligned}$$

where the second equality is because if z is normally distributed with mean zero, then

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}z^2\right\}\right] = (1 + \mathrm{Var}(z))^{-\frac{1}{2}}.$$

We now show the proposition. Note that $\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), \boldsymbol{y}_i, p_2) = \mathbb{E}(\theta|\boldsymbol{y}_i, p_2)$. Suppose that the two economies are not equivalent. We first show, by contradiction, that $\mathbb{E}(\theta|\boldsymbol{y}_i, p_2) \neq \mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p_2)$ for at least one i. Indeed, if

$$\mathbb{E}(\theta|\boldsymbol{y}_i, p_2) = \mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p_2)$$

for every i, then

$$\mathrm{Var}(\theta|\boldsymbol{y}_i, p_2) = \mathrm{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p_2)$$

by the Law of Total Variance for every i, implying that p_2 is also an equilibrium price of **Economy 1** because the market-clearing condition in **Economy 1** also holds by replacing p_1 with p_2. Because we have assumed in this study that the linear equilibrium is unique, $p_2 = p_1$ almost surely, denoted as p. As a result, $\mathbb{E}(\theta|\boldsymbol{y}_i, p) = \mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p)$ almost surely for every i and the two economies are thus equivalent by Proposition 2, a contradiction. Therefore, $\mathbb{E}(\theta|\boldsymbol{y}_i, p_2) \neq \mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p_2)$ for some i. It then follows from the Law of Total Variance that $\mathrm{Var}(\theta|\boldsymbol{y}_i, p_2) < \mathrm{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p_2)$. The conclusion follows from (A.7). □

Proof of Proposition 2. We first show the necessity. By virtue of (1), we have

$$x^*_{i,1} = \frac{\mathbb{E}(\theta|\mathcal{F}^1_i) - p}{\rho_i \mathrm{Var}(\theta|\mathcal{F}^1_i)}, \quad \text{(A.8)}$$

where p is the (common) equilibrium price of the two economies. As $p \in \mathcal{F}^1_i$ and the conditional variance $\mathrm{Var}(\theta|\mathcal{F}^1_i)$ is a constant under normality settings, by the *tower* property of conditional expectation we obtain

$$\mathbb{E}(x^*_{i,1}|p) = \frac{\mathbb{E}(\theta|p) - p}{\rho_i \mathrm{Var}(\theta|\mathcal{F}^1_i)}. \quad \text{(A.9)}$$

A similar expression can be given for the conditional expectation $\mathbb{E}(x^*_{i,2}|p)$ by replacing $\mathrm{Var}(\theta|\mathcal{F}^1_i)$ with $\mathrm{Var}(\theta|\mathcal{F}^2_i)$ in (A.9). Since $x^*_{i,1} = x^*_{i,2}$ almost surely, we have from (A.9) that $\mathrm{Var}(\theta|\mathcal{F}^1_i) = \mathrm{Var}(\theta|\mathcal{F}^2_i)$,[23] and further from (A.8) that $\mathbb{E}(\theta|\mathcal{F}^1_i) = \mathbb{E}(\theta|\mathcal{F}^2_i)$, i.e., $\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p) = \mathbb{E}(\theta|\boldsymbol{y}_i, p)$ almost surely for every i. The necessity follows.

Now, we show the sufficiency. It follows from the Law of Total Variance $\mathrm{Var}(\theta) = \mathrm{Var}(\mathbb{E}(\theta|\cdot)) + \mathbb{E}(\mathrm{Var}(\theta|\cdot))$ and the sufficiency condition $\mathbb{E}(\theta|\boldsymbol{y}_i, p) = \mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p)$ that $\mathrm{Var}(\theta|\boldsymbol{y}_i, p) = \mathrm{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p)$. The preceding two equalities and the market-clearing condition imply that when p is an equilibrium price of one of the two economies, it is also the equilibrium price of the other one. Hence, $x^i_{av} = x^i_{di}$ almost surely, by the optimal demand (1), implying the equivalence. □

Proof of Theorem 1. By virtue of Proposition 2, it suffices to show that

$$p = \mathbb{E}(\theta|\boldsymbol{y}) + \frac{n\,\mathrm{Var}(\theta|\boldsymbol{y})}{\sum_{i=1}^n \frac{1}{\rho_i}}u$$

is an equilibrium price in **Economy 2** and satisfies that

$$\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p) = \mathbb{E}(\theta|\boldsymbol{y}_i, p)$$

almost surely for every i. Indeed, when the network graph is complete, $\boldsymbol{y}_i = \boldsymbol{y}$ for every i. Then for the price

$$p = \mathbb{E}(\theta|\boldsymbol{y}) + \frac{n\,\mathrm{Var}(\theta|\boldsymbol{y})}{\sum_{i=1}^n \frac{1}{\rho_i}}u,$$

it clearly holds that

$$\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}), p) = \mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}), u) = \mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y})) = \mathbb{E}(\theta|\boldsymbol{y})$$

and $\mathbb{E}(\theta|\boldsymbol{y}, p) = \mathbb{E}(\theta|\boldsymbol{y}, u) = \mathbb{E}(\theta|\boldsymbol{y})$ by the independence of u and $\boldsymbol{y}$, and Lemma 2. This price also satisfies the market-clearing condition in **Economy 2**, noting that

[23] We thank the referee for suggesting this simple method. An alternative approach is to apply formula (A.7).

$$\sum_{i=1}^{n} \frac{\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}), p) - p}{\rho_i \operatorname{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}), p)} + nu = \sum_{i=1}^{n} \frac{\mathbb{E}(\theta|\boldsymbol{y}) - p}{\rho_i \operatorname{Var}(\theta|\boldsymbol{y})} + nu = 0,$$

where we use the equality $\operatorname{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}), p) = \operatorname{Var}(\theta|\boldsymbol{y})$, which follows from the Law of Total Variance and the relation $\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}), p) = \mathbb{E}(\theta|\boldsymbol{y})$ that we proved above. The proof is complete. □

Proof of Proposition 3. Suppose the two economies are equivalent and let $p = \boldsymbol{\pi}'\boldsymbol{y} + \gamma u$ be the same equilibrium price of the two economies. Denote $\mathcal{S} = \{i|\mathcal{N}_i \neq \{1, \dots, n\}\}$. As the network graph is not complete, $\mathcal{S} \neq \emptyset$. By Proposition 2, $\mathbb{E}(\theta|\boldsymbol{y}_i, p) = \mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p)$ almost surely for every i. Hence, if there exists $i \in \mathcal{S}$ such that $\mathbf{Cov}(\theta, \boldsymbol{y}_i) \neq c\mathbf{Cov}(p, \boldsymbol{y}_i)$ for any $c \neq 0$, then invoking Lemma 5, we obtain

$$\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p) = \mathbb{E}(\theta|\boldsymbol{y}_i, p) = \mathbb{E}(\theta|\boldsymbol{y}_i) = \mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i))$$

almost surely, where the last equality follows from Lemma 2. The Law of Total Variance then indicates that $\operatorname{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p) = \operatorname{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i))$, contradicting Assumption 2.

Thus, for any $i \in \mathcal{S}$, $\mathbf{Cov}(\theta, \boldsymbol{y}_i) = c_i\mathbf{Cov}(p, \boldsymbol{y}_i)$, or equivalently,

$$\mathbf{Cov}(\theta, \boldsymbol{y}_i) = \mathbf{Cov}(\boldsymbol{\pi}'\boldsymbol{y}/c_i, \boldsymbol{y}_i)$$

for some $c_i \neq 0$. Observe that $\mathcal{N}_j = \{1, \dots, n\}$ for any $j \in \{1, \dots, n\}/\mathcal{S}$. Noticing that $\operatorname{Cov}(\theta, y_i) > 0$ for every i (Assumption 1), and the network graph is strongly connected, we have

$$\mathbf{Cov}(\theta, \boldsymbol{y}) = \mathbf{Cov}(\boldsymbol{\pi}'\boldsymbol{y}/c, \boldsymbol{y})$$

for some $c \neq 0$. As a result,

$$\boldsymbol{\pi}/c = \mathbf{Var}(\boldsymbol{y})^{-1}\mathbf{Cov}(\theta, \boldsymbol{y}).$$

That is, for some $c \neq 0$,

$$p = \boldsymbol{\pi}'\boldsymbol{y} + \gamma u = c\mathbf{Cov}(\theta, \boldsymbol{y})'\mathbf{Var}(\boldsymbol{y})^{-1}\boldsymbol{y} + \gamma u = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u,$$

where we use the fact $\mathbb{E}(\theta|\boldsymbol{y}) = \mathbf{Cov}(\theta, \boldsymbol{y})'\mathbf{Var}(\boldsymbol{y})^{-1}\boldsymbol{y}$ (Lemma 1).

It remains to show that $c > 0$ and $\gamma > 0$. Observe that

$$\operatorname{Cov}(\theta, \mathbb{E}(\theta|\boldsymbol{y}_i)) = \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) = \mathbf{Cov}(\theta, \boldsymbol{y}_i)'\mathbf{Var}(\boldsymbol{y}_i)^{-1}\mathbf{Cov}(\theta, \boldsymbol{y}_i) > 0$$

for every i, since $\mathbf{Cov}(\theta, \boldsymbol{y}_i) \neq \mathbf{0}$ by Assumption 1, where the first equality follows from Lemma 3 (i), and the second from $\mathbb{E}(\theta|\boldsymbol{y}_i) = \mathbf{Cov}(\theta, \boldsymbol{y}_i)'\mathbf{Var}(\boldsymbol{y}_i)^{-1}\boldsymbol{y}_i$ (Lemma 1). With the preceding relation, it follows from the independence of u and the other random variables and Proposition 5 in Lou et al. (2019) that

$$\operatorname{Cov}(\theta, p) = c\operatorname{Cov}(\theta, \mathbb{E}(\theta|\boldsymbol{y})) = c\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y})) > 0.$$

Observe that

$$\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y})) = \mathbf{Cov}(\theta, \boldsymbol{y})'\mathbf{Var}(\boldsymbol{y})^{-1}\mathbf{Cov}(\theta, \boldsymbol{y}) > 0$$

since $\mathbf{Cov}(\theta, \boldsymbol{y}) \neq \mathbf{0}$ by Assumption 1. The two preceding relations imply that $c > 0$. Furthermore, Proposition 2 in Lou et al. (2019) also informs us that $\gamma > 0$. The proof is complete. □

Proof of Proposition 4. This relation clearly holds for any i with $\mathcal{N}_i = \{1, \dots, n\}$ (see the arguments at the end of the proof of Theorem 1). We next consider the index i with $\mathcal{N}_i \neq \{1, \dots, n\}$. Observing (A.1), (A.2) and (A.3) with the identity $z = p = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u$ and $\boldsymbol{s} = \boldsymbol{y}_i$, it is observed that $\mathbb{E}(\theta|\boldsymbol{y}_i, p) = \mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i), p)$ almost surely if and only if the following two equalities hold:

$$\frac{\operatorname{Cov}(p, \theta) - \mathbf{Cov}(\theta, \boldsymbol{y}_i)'\mathbf{Var}(\boldsymbol{y}_i)^{-1}\mathbf{Cov}(p, \boldsymbol{y}_i)}{\operatorname{Var}(p) - \mathbf{Cov}(p, \boldsymbol{y}_i)'\mathbf{Var}(\boldsymbol{y}_i)^{-1}\mathbf{Cov}(p, \boldsymbol{y}_i)} = \frac{\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))\operatorname{Cov}(\theta, p) - \operatorname{Cov}(\theta, \mathbb{E}(\theta|\boldsymbol{y}_i))\operatorname{Cov}(\mathbb{E}(\theta|\boldsymbol{y}_i), p)}{\operatorname{Var}(p)\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) - \operatorname{Cov}(\mathbb{E}(\theta|\boldsymbol{y}_i), p)^2}, \quad \text{(A.10)}$$

$$\left[\mathbf{Var}(\boldsymbol{y}_i) - \frac{\mathbf{Cov}(p, \boldsymbol{y}_i)\mathbf{Cov}(p, \boldsymbol{y}_i)'}{\operatorname{Var}(p)}\right]^{-1}\left[\mathbf{Cov}(\theta, \boldsymbol{y}_i) - \frac{\operatorname{Cov}(p, \theta)}{\operatorname{Var}(p)}\mathbf{Cov}(p, \boldsymbol{y}_i)\right]$$
$$= \frac{\operatorname{Var}(p)\operatorname{Cov}(\theta, \mathbb{E}(\theta|\boldsymbol{y}_i)) - \operatorname{Cov}(\mathbb{E}(\theta|\boldsymbol{y}_i), p)\operatorname{Cov}(\theta, p)}{\operatorname{Var}(p)\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) - \operatorname{Cov}(\mathbb{E}(\theta|\boldsymbol{y}_i), p)^2}\mathbf{Var}(\boldsymbol{y}_i)^{-1}\mathbf{Cov}(\theta, \boldsymbol{y}_i), \quad \text{(A.11)}$$

where we use the fact $\mathbb{E}(\theta|\boldsymbol{y}_i) = \mathbf{Cov}(\theta, \boldsymbol{y}_i)'\mathbf{Var}(\boldsymbol{y}_i)^{-1}\boldsymbol{y}_i$ (Lemma 1).

We first consider (A.10). Observe the following series of relations:

$$\mathbf{Cov}(\theta, \boldsymbol{y}_i)'\mathbf{Var}(\boldsymbol{y}_i)^{-1}\mathbf{Cov}(p, \boldsymbol{y}_i) = \operatorname{Cov}(p, \mathbb{E}(\theta|\boldsymbol{y}_i)) \text{ (Lemma 1)},$$
$$\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) = \operatorname{Cov}(\theta, \mathbb{E}(\theta|\boldsymbol{y}_i)) \text{ (Lemma 3 (i))}, \quad \text{(A.12)}$$

$$\operatorname{Cov}(\mathbb{E}(\theta|\boldsymbol{y}_i), p) = c\operatorname{Cov}(\mathbb{E}(\theta|\boldsymbol{y}_i), \mathbb{E}(\theta|\boldsymbol{y})) = c\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) \text{ (Lemma 3 (ii))}, \quad \text{(A.13)}$$

$$\operatorname{Cov}(\theta, p) = c\operatorname{Cov}(\theta, \mathbb{E}(\theta|\boldsymbol{y})) = c\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y})) \text{ (Lemma 3 (i))}, \quad \text{(A.14)}$$

and $\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y})) > \operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))$ (see Remark 1 for the derivation). We then have $\operatorname{Cov}(\theta, p) \neq \operatorname{Cov}(p, \mathbb{E}(\theta|\boldsymbol{y}_i))$. Consequently, (A.10) is equivalent to

$$\frac{1}{\operatorname{Var}(p) - \mathbf{Cov}(p, \boldsymbol{y}_i)'\mathbf{Var}(\boldsymbol{y}_i)^{-1}\mathbf{Cov}(p, \boldsymbol{y}_i)} = \frac{\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))}{\operatorname{Var}(p)\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) - \operatorname{Cov}(\mathbb{E}(\theta|\boldsymbol{y}_i), p)^2},$$

or further equivalent to

$$\mathbf{Cov}(p, \boldsymbol{y}_i)'\mathbf{Var}(\boldsymbol{y}_i)^{-1}\mathbf{Cov}(p, \boldsymbol{y}_i)\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) = \operatorname{Cov}(\mathbb{E}(\theta|\boldsymbol{y}_i), p)^2. \quad \text{(A.15)}$$

By Lemma 3 (iii), we have

$$\mathbf{Cov}(p, \boldsymbol{y}_i) = c\mathbf{Cov}(\mathbb{E}(\theta|\boldsymbol{y}), \boldsymbol{y}_i) = c\mathbf{Cov}(\theta, \boldsymbol{y}_i). \quad \text{(A.16)}$$

It is easily verified that (A.15) indeed holds, noting (A.13) and the equality

$$\mathbf{Cov}(p, \boldsymbol{y}_i)'\mathbf{Var}(\boldsymbol{y}_i)^{-1}\mathbf{Cov}(p, \boldsymbol{y}_i) = c^2\mathbf{Cov}(\theta, \boldsymbol{y}_i)'\mathbf{Var}(\boldsymbol{y}_i)^{-1}\mathbf{Cov}(\theta, \boldsymbol{y}_i) = c^2\mathbf{Cov}(\theta, \mathbb{E}(\theta|\boldsymbol{y}_i)) = c^2\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)),$$

where the first equality follows from (A.16), the second from the expression $\mathbb{E}(\theta|\boldsymbol{y}_i) = \mathbf{Cov}(\theta, \boldsymbol{y}_i)'\mathbf{Var}(\boldsymbol{y}_i)^{-1}\boldsymbol{y}_i$ (Lemma 1), and the third from (A.12). As a summary, the above analysis shows that for any $c > 0$ and $\gamma > 0$, the price function $p = c\mathbb{E}(\theta|\boldsymbol{y}) + \gamma u$ satisfies (A.10).

We now turn to (A.11). Noting (A.12), we see that (A.11) is equivalent to

$$\mathbf{Cov}(\theta, \boldsymbol{y}_i) - \frac{\operatorname{Cov}(p, \theta)}{\operatorname{Var}(p)}\mathbf{Cov}(p, \boldsymbol{y}_i)$$
$$= \frac{\operatorname{Var}(p)\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) - \operatorname{Cov}(\mathbb{E}(\theta|\boldsymbol{y}_i), p)\operatorname{Cov}(\theta, p)}{\operatorname{Var}(p)\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) - \operatorname{Cov}(\mathbb{E}(\theta|\boldsymbol{y}_i), p)^2} \times \left[\boldsymbol{I}_m - \frac{\mathbf{Cov}(p, \boldsymbol{y}_i)\mathbf{Cov}(p, \boldsymbol{y}_i)'}{\operatorname{Var}(p)}\mathbf{Var}(\boldsymbol{y}_i)^{-1}\right]\mathbf{Cov}(\theta, \boldsymbol{y}_i)$$
$$= \frac{\operatorname{Var}(p)\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) - \operatorname{Cov}(\mathbb{E}(\theta|\boldsymbol{y}_i), p)\operatorname{Cov}(\theta, p)}{\operatorname{Var}(p)\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i)) - \operatorname{Cov}(\mathbb{E}(\theta|\boldsymbol{y}_i), p)^2} \times \left[\mathbf{Cov}(\theta, \boldsymbol{y}_i) - \frac{\operatorname{Cov}(p, \mathbb{E}(\theta|\boldsymbol{y}_i))}{\operatorname{Var}(p)}\mathbf{Cov}(p, \boldsymbol{y}_i)\right]. \quad \text{(A.17)}$$

With (A.16), (A.14), and (A.13), it is easy to see that (A.17) is further equivalent to

$$\left[1-\frac{\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}))}{\mathrm{Var}(p)}c^2\right]\mathbf{Cov}(\theta,\boldsymbol{y}_i)$$
$$=\frac{\mathrm{Var}(p)\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))-c^2\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}))}{\mathrm{Var}(p)\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))-c^2\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))^2}\times\left[1-\frac{\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))}{\mathrm{Var}(p)}c^2\right]\mathbf{Cov}(\theta,\boldsymbol{y}_i).$$

Because $\mathbf{Cov}(\theta,\boldsymbol{y}_i)\neq\mathbf{0}$ (Assumption 1), the preceding equality is also equivalent to

$$1-\frac{\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}))c^2}{\mathrm{Var}(p)}=\frac{\mathrm{Var}(p)\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))-c^2\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}))}{\mathrm{Var}(p)\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))-c^2\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))^2}\times\left[1-\frac{\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))c^2}{\mathrm{Var}(p)}\right].$$

Now, with $\mathrm{Var}(p)=c^2\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}))+\gamma^2\,\mathrm{Var}(u)$, it can be verified that the preceding equality is always true. That is, for any $c>0$ and $\gamma>0$, (A.11) holds for the price function $p=c\mathbb{E}(\theta|\boldsymbol{y})+\gamma u$. The proof is complete. □

Proof Theorem 3. Note that there exists at least one index, say i, such that the weight given to signal y_i in $\mathbb{E}(\theta|\boldsymbol{y})$ is nonzero because, otherwise, $\mathbb{E}(\theta|\boldsymbol{y})=0$ almost surely, contradicting Assumption 1. For this signal structure, we construct a network graph as follows: Select one index $j\neq i$, and let $\mathcal{N}_j=\{1,\dots,n\}/\{i\}$ and $\mathcal{N}_r=\{1,\dots,n\}$ for every $r\neq j$. That is, this graph includes all possible arcs except for the arc from i to j.

From the market-clearing condition

$$\sum_{i=1}^{n}\frac{\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i),p)-p}{\rho_i\,\mathrm{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i),p)}+nu$$
$$=\frac{\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_j),p)-p}{\rho_j\,\mathrm{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_j),p)}+\sum_{r\neq j}\frac{\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}),p)-p}{\rho_r\,\mathrm{Var}(\theta|\mathbb{E}(\theta|\boldsymbol{y}),p)}+nu$$
$$=0,\tag{A.18}$$

it can be seen that the equilibrium price can be expressed as a linear form $p=b_1\mathbb{E}(\theta|\boldsymbol{y}_j)+b_2\mathbb{E}(\theta|\boldsymbol{y})+\gamma u$, where b_1, b_2, and γ are constants. We can show by contradiction that $b_2\neq 0$. Otherwise, suppose $p=b_1\mathbb{E}(\theta|\boldsymbol{y}_j)+\gamma u$. From (A.3), with the identity $z=p$ and $\boldsymbol{s}=\boldsymbol{y}$, we see that the coefficient on $\mathbb{E}(\theta|\boldsymbol{y})$ in $\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}),p)$ equals

$$\frac{\mathrm{Var}(p)\,\mathrm{Cov}(\theta,\mathbb{E}(\theta|\boldsymbol{y}))-\mathrm{Cov}(\mathbb{E}(\theta|\boldsymbol{y}),p)\,\mathrm{Cov}(\theta,p)}{\mathrm{Var}(p)\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}))-\mathrm{Cov}(\mathbb{E}(\theta|\boldsymbol{y}),p)^2}.\tag{A.19}$$

Noting the series of relations:

$$\mathrm{Var}(p)=b_1^2\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_j))+\gamma^2\,\mathrm{Var}(u),$$
$$\mathrm{Cov}(\theta,\mathbb{E}(\theta|\boldsymbol{y}))=\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}))\ \text{(Lemma 3 (i))},$$
$$\mathrm{Cov}(\mathbb{E}(\theta|\boldsymbol{y}),p)=b_1\,\mathrm{Cov}(\mathbb{E}(\theta|\boldsymbol{y}),\mathbb{E}(\theta|\boldsymbol{y}_j))=b_1\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_j))\ \text{(Lemma 3 (ii))},\tag{A.20}$$
$$\mathrm{Cov}(\theta,p)=b_1\,\mathrm{Cov}(\theta,\mathbb{E}(\theta|\boldsymbol{y}_j))=b_1\,\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_j))\ \text{(Lemma 3 (i))}\tag{A.21}$$

and $\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_j))<\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}))$ (see the arguments about the inequality in Remark 1), the term in (A.19) does not equal zero. As a result, the market-clearing condition (A.18) is impossible to hold for the price function $p=b_1\mathbb{E}(\theta|\boldsymbol{y}_j)+\gamma u$. Hence, $b_2\neq 0$.

For the price function $p=b_1\mathbb{E}(\theta|\boldsymbol{y}_j)+b_2\mathbb{E}(\theta|\boldsymbol{y})+\gamma u$, it holds that

$$\mathrm{Cov}(\theta-\mathbb{E}(\theta|\boldsymbol{y}_j),p)=b_2[\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}))-\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_j))]\neq 0,$$

where the equality uses (A.20) and (A.21). It then follows from Lemma 4 that Assumption 2 holds. However, for the given signal structure and the accordingly constructed network graph, (6) is impossible to hold for any positive γ and c. The two economies are thus not equivalent under the constructed graph by Theorem 2. The proof is complete. □

Proof Theorem 4. We assume that all signals are normally distributed with means zero and consider the following signal structure: $\mathrm{Var}(y_i)=1$, $\mathrm{Cov}(\theta,y_i)>0$ for every i but

$$\sum_{i=1}^{n}\mathrm{Cov}(\theta,y_i)^2<\mathrm{Var}(\theta),$$

and for every i, y_i is independent of $\{y_j,j\neq i\}$. Under this signal structure,

$$\mathbb{E}(\theta|\boldsymbol{y})=\sum_{i=1}^{n}\mathrm{Cov}(\theta,y_i)y_i,\quad \mathbb{E}(\theta|\boldsymbol{y}_i)=\sum_{j\in\mathcal{N}_i}\mathrm{Cov}(\theta,y_j)y_j.$$

The preceding two equalities imply that the strict inequality assumption in Assumption 1 holds. We also see that the variance–covariance matrix of random vector $(\theta,\boldsymbol{y}')'$ is positive definite because, otherwise, on the one hand it follows from the independence of $\{y_i,i=1,\dots,n\}$ that $\mathrm{Var}(\theta|\boldsymbol{y})=0$, but, on the other hand,

$$\mathrm{Var}(\theta|\boldsymbol{y})=\mathrm{Var}(\theta)-\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}))=\mathrm{Var}(\theta)-\sum_{i=1}^{n}\mathrm{Cov}(\theta,y_i)^2>0,$$

a contradiction. Hence, Assumption 1 holds.

We complete the proof by contradiction. Suppose that the two economies are equivalent and let $p=\boldsymbol{\pi}'\boldsymbol{y}+\gamma u$ denote the common equilibrium price. Proposition 7 (i) in Lou ⓡ al. (2019) shows that the weight on each signal y_i in the equilibrium price p is positive, that is, $\pi_i>0$ for every i. We have

$$\mathrm{Cov}(\theta,p)=\sum_{i=1}^{n}\pi_i\,\mathrm{Cov}(\theta,y_i),$$

and for any i with $\mathcal{N}_i\neq\{1,\dots,n\}$,

$$\mathrm{Cov}(\mathbb{E}(\theta|\boldsymbol{y}_i),p)=\mathrm{Cov}\Big(\sum_{j\in\mathcal{N}_i}\mathrm{Cov}(\theta,y_j)y_j,p\Big)=\sum_{j\in\mathcal{N}_i}\pi_j\,\mathrm{Cov}(\theta,y_j),$$

implying that, together with Lemma 4, Assumption 2 holds. Furthermore, by the equivalence assumption and Theorem 2, there exist $c>0$ and $\gamma>0$ such that (6) holds, i.e.,

$$\gamma\sum_{i=1}^{n}\frac{\gamma^2\,\mathrm{Var}(u)}{\rho_i\{\gamma^2\,\mathrm{Var}(u)a_i+[\gamma^2\,\mathrm{Var}(u)+c^2a_i][\mathrm{Var}(\theta)-\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}))]\}}\times\sum_{j\in\mathcal{N}_i}\mathrm{Cov}(\theta,y_j)y_j=c\sum_{i=1}^{n}\mathrm{Cov}(\theta,y_i)y_i$$

almost surely. Therefore, on the one hand, by matching the coefficients on both sides of the preceding equality, we see that the scalar

$$\sum_{\{j|i\in\mathcal{N}_j\}}\frac{1}{\rho_j\{\gamma^2\,\mathrm{Var}(u)a_j+[\gamma^2\,\mathrm{Var}(u)+c^2a_j][\mathrm{Var}(\theta)-\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}))]\}}\tag{A.22}$$

does not vary over the index i, where

$$a_j=\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}))-\mathrm{Var}(\mathbb{E}(\theta|\boldsymbol{y}_j))=\sum_{r\notin\mathcal{N}_j}\mathrm{Cov}(\theta,y_r)^2.$$

$$\frac{\gamma}{n\rho}\sum_{i=1}^{n}\frac{\gamma^2\operatorname{Var}(u)+(c^2-c)(\frac{1}{\tau_\theta+|\mathcal{N}_i|\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon})}{[\gamma^2\operatorname{Var}(u)\frac{1}{\tau_\theta+|\mathcal{N}_i|\tau_\epsilon}+c^2(\frac{1}{\tau_\theta+|\mathcal{N}_i|\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon})]\frac{1}{\tau_\theta+|\mathcal{N}_i|\tau_\epsilon}-c^2(\frac{1}{\tau_\theta+|\mathcal{N}_i|\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon})^2}=1,$$

Box II.

$$\frac{\gamma^3\operatorname{Var}(u)}{cn\rho}\sum_{\{j|i\in\mathcal{N}_j\}}\frac{\frac{\tau_\epsilon}{\tau_\theta+|\mathcal{N}_j|\tau_\epsilon}}{[\gamma^2\operatorname{Var}(u)+c^2(\frac{1}{\tau_\theta+|\mathcal{N}_j|\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon})]\frac{1}{\tau_\theta+|\mathcal{N}_j|\tau_\epsilon}-c^2(\frac{1}{\tau_\theta+|\mathcal{N}_j|\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon})^2}=\frac{\tau_\epsilon}{\tau_\theta+n\tau_\epsilon},$$

Box III.

$$\frac{\gamma^3\operatorname{Var}(u)}{cn\rho}\sum_{\{j|i\in\mathcal{N}_j\}}\frac{\frac{1}{\tau_\theta+|\mathcal{N}_j|\tau_\epsilon}}{[\gamma^2\operatorname{Var}(u)+c^2(\frac{1}{\tau_\theta+|\mathcal{N}_j|\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon})]\frac{1}{\tau_\theta+|\mathcal{N}_j|\tau_\epsilon}-c^2(\frac{1}{\tau_\theta+|\mathcal{N}_j|\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon})^2}=\frac{1}{\tau_\theta+n\tau_\epsilon},$$

Box IV.

Note that for any non-complete graph, there always exist two nodes which have different sets of out-neighbors, say, i and j are two nodes such that $\{r|i\in\mathcal{N}_r\}\neq\{r|j\in\mathcal{N}_r\}$. However, on the other hand, we can select the values for $\{\operatorname{Cov}(\theta,y_i)\}$ such that (A.22) does not equal for nodes i and j, raising a contradiction. The proof is complete. □

Proof Corollary 1. Denote $\tau_\theta=\frac{1}{\operatorname{Var}(\theta)}$ and $\tau_\epsilon=\frac{1}{\operatorname{Var}(\epsilon_i)}$. Under the signal structure in Assumption 3,

$$\operatorname{Var}(\theta|\boldsymbol{y}_i)=\frac{1}{\tau_\theta+|\mathcal{N}_i|\tau_\epsilon},\quad \mathbb{E}(\theta|\boldsymbol{y}_i)=\frac{\tau_\epsilon}{\tau_\theta+|\mathcal{N}_i|\tau_\epsilon}\sum_{j\in\mathcal{N}_i}y_j$$

and

$$a_i=\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}))-\operatorname{Var}(\mathbb{E}(\theta|\boldsymbol{y}_i))=\frac{1}{\tau_\theta+|\mathcal{N}_i|\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon}.$$

Then (5) and (6) respectively reduce to the equation given in Box II, which is exactly (7) by some simple computations, and

$$\frac{\gamma^3\operatorname{Var}(u)}{cn\rho}\sum_{i=1}^{n}\frac{\frac{\tau_\epsilon}{\tau_\theta+|\mathcal{N}_i|\tau_\epsilon}\sum_{j\in\mathcal{N}_i}y_j}{[\gamma^2\operatorname{Var}(u)+c^2a_i]\frac{1}{\tau_\theta+|\mathcal{N}_i|\tau_\epsilon}-c^2a_i^2}$$
$$=\frac{\tau_\epsilon}{\tau_\theta+n\tau_\epsilon}\sum_{i=1}^{n}y_i \text{ almost surely.}$$

Matching the coefficients in the preceding equality leads to the case that for every i, as given in Box III, or, equivalently as in Box IV, which is exactly (8) by some simple computations. □

Proof Corollary 2. When the network graph is k-regular, that is, $|\mathcal{N}_i|=k$ for every i, Eqs. (7) and (8) respectively reduce to

$$\frac{\gamma}{\rho}\frac{\gamma^2\operatorname{Var}(u)+(c^2-c)(\frac{1}{\tau_\theta+k\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon})}{\gamma^2\operatorname{Var}(u)\frac{1}{\tau_\theta+k\tau_\epsilon}+c^2(\frac{1}{\tau_\theta+k\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon})\frac{1}{\tau_\theta+n\tau_\epsilon}}=1,$$
$$\frac{k\gamma^3\operatorname{Var}(u)}{cn\rho}\frac{\tau_\theta+n\tau_\epsilon}{\gamma^2\operatorname{Var}(u)+c^2\frac{(n-k)\tau_\epsilon}{(\tau_\theta+n\tau_\epsilon)^2}}=1.$$

Equivalently, letting $1/\gamma=\delta$, the preceding two equations can be respectively rewritten as

$$\frac{1}{\rho}\frac{\operatorname{Var}(u)+\delta^2(c^2-c)(\frac{1}{\tau_\theta+k\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon})}{\operatorname{Var}(u)\frac{1}{\tau_\theta+k\tau_\epsilon}+\delta^2c^2(\frac{1}{\tau_\theta+k\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon})\frac{1}{\tau_\theta+n\tau_\epsilon}}=\delta, \tag{A.23}$$
$$\frac{k\operatorname{Var}(u)}{n\rho}\frac{\tau_\theta+n\tau_\epsilon}{\operatorname{Var}(u)+\delta^2c^2\frac{(n-k)\tau_\epsilon}{(\tau_\theta+n\tau_\epsilon)^2}}=\delta c. \tag{A.24}$$

We obtain a positive solution for δc, denoted a_*, from Eq. (A.24), considering that it is a cubic equation of the variable δc. It follows that

$$\operatorname{Var}(u)+(a_*)^2\frac{(n-k)\tau_\epsilon}{(\tau_\theta+n\tau_\epsilon)^2}=\frac{k\operatorname{Var}(u)(\tau_\theta+n\tau_\epsilon)}{n\rho a_*}.$$

Substituting the preceding expression into Eq. (A.23) yields

$$\frac{\operatorname{Var}(u)+\delta^2(c^2-c)(\frac{1}{\tau_\theta+k\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon})}{\frac{k(\tau_\theta+n\tau_\epsilon)}{n(\tau_\theta+k\tau_\epsilon)}\frac{\operatorname{Var}(u)}{a_*}}=\delta.$$

Multiplying both sides of the preceding equation by c, with $a_*=\delta c$ in mind, leads to

$$\frac{c[\operatorname{Var}(u)+a_*^2(\frac{1}{\tau_\theta+k\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon})]-a_*^2(\frac{1}{\tau_\theta+k\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon})}{\frac{k(\tau_\theta+n\tau_\epsilon)}{n(\tau_\theta+k\tau_\epsilon)}\frac{\operatorname{Var}(u)}{a_*}}=a_*,$$

from which we obtain a solution for c:

$$c_*=\frac{\frac{k(\tau_\theta+n\tau_\epsilon)}{n(\tau_\theta+k\tau_\epsilon)}\operatorname{Var}(u)+a_*^2(\frac{1}{\tau_\theta+k\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon})}{\operatorname{Var}(u)+a_*^2(\frac{1}{\tau_\theta+k\tau_\epsilon}-\frac{1}{\tau_\theta+n\tau_\epsilon})}>0.$$

As a consequence, we obtain a positive solution $\delta_*=a_*/c_*$ and then $\gamma_*=1/\delta_*$.

The non-equivalence between the two economies for chain and star graphs can be seen by noting that the term in the summation of (8) in Corollary 1 is monotonically increasing in $|\mathcal{N}_j|$, and then (8) is impossible to hold for every i. The proof is completed. □

Proof Theorem 5. Let $p=c\theta+\gamma u$ be the price function, where the two constants c and γ will be determined by the market-clearing condition shown below. Consider some independent subnetwork in an m-replica economy as $\mathcal{A}_m$. The total demand of the n traders in the subnetwork at price p is given by

$$\sum_{i=1}^{n}\frac{\mathbb{E}(\theta|\boldsymbol{y}_i)-p}{\rho\,\mathrm{Var}(\theta|\boldsymbol{y}_i)}=\sum_{i=1}^{n}\frac{\sum_{j\in\mathcal{N}_i}\tau_{\epsilon_j}y_j}{\rho}-\sum_{i=1}^{n}\frac{\tau_\theta+\sum_{j\in\mathcal{N}_i}\tau_{\epsilon_j}}{\rho}p$$
$$=\frac{1}{\rho}\left(\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i}\tau_{\epsilon_j}(\theta+\epsilon_j)-\sum_{i=1}^{n}\left(\tau_\theta+\sum_{j\in\mathcal{N}_i}\tau_{\epsilon_j}\right)p\right).$$

Now, consider the large economy $\mathcal{A}$, which is the limit of the sequence of $\{\mathcal{A}_m\}$. As noise terms are mutually independent across traders and the replica subnetworks have an identical network structure, the noise will disappear in the average of the total demand of traders in the large economy by applying the Law of Large Numbers. Therefore, the price given in the theorem satisfies the market-clearing condition in the large economy. By Lemma 1, with some simple computations, the price given in this theorem also satisfies the condition $\mathbb{E}(\theta|\boldsymbol{y}_i,p)=\mathbb{E}(\theta|\mathbb{E}(\theta|\boldsymbol{y}_i),p)$ almost surely for every i. It thus follows from Proposition 2 that the two large economies $\mathcal{A}^1$ and $\mathcal{A}^2$ are equivalent. This completes the proof. □

References

Banerjee, A., 1992. A simple model of herd behavior. Q. J. Econ. 107, 797–817.
Bikhchandani, S., Hirshleifer, D., Welch, I., 1992. A theory of fads, fashion, custom, and cultural change as information cascades. J. Polit. Econ. 100, 992–1026.
Brown, J.R., Ivkovich, Z., Smith, P.A., Weisbenner, S., 2008. Neighbors matter: Causal community effects and stock market participation. J. Finance 63, 1509–1531.
Carpio, R., Guo, M., 2019. On equilibrium existence in a finite-agent, multi- asset noisy rational expectations economy. B.E. J. Theor. Econ. 20180144.
Colla, P., Mele, A., 2010. Information linkages and correlated trading. Rev. Financ. Stud. 23, 203–246.
DeGroot, M.H., 1970. Optimal Statistical Decisions. McGraw-Hall, New York.
DeGroot, M.H., 1974. Reaching a consensus. J. Amer. Statist. Assoc. 69, 118–121.
Demarzo, P.M., Vayanos, D., Zwiebel, J., 2003. Persuasion bias, social influence, and unidimensional opinions. Q. J. Econ. 118, 909–968.
Feng, L., Seasholes, M.S., 2004. Correlated trading and location. J. Finance 59, 2117–2144.
Ganguli, J.V., Yang, L., 2009. Complementarities, multiplicity, and supply information. J. Eur. Econom. Assoc. 7, 90–115.
Goldstein, I., Yang, L., 2015. Information diversity and complementarities in trading and information acquisition. J. Finance 70, 1723–1765.
Goldstein, I., Yang, L., 2017. Information disclosure in financial markets. Ann. Rev. Financ. Econ. 9, 101–125.
Grossman, S., 1976. On the efficiency of competitive stock markets where traders have diverse information. J. Finance 31, 573–585.
Grossman, S.J., Stiglitz, J.E., 1980. On the impossibility of informationally efficient markets. Amer. Econ. Rev. 70, 393–408.
Halim, E., Riyanto, Y.E., Roy, N., 2019. Costly information acquisition, social networks, and asset prices: Experimental evidence. J. Finance 74, 1975–2010.
Han, B., Yang, L., 2013. Social networks, information acquisition and asset prices. Manag. Sci. 59, 1444–1457.
He, X.Z., Shi, L., Tolotti, M., 2019. Welfare analysis under probabilistic choices in a rational expectations equilibrium model. Working paper.
Heimer, R.Z., 2016. Peer pressure: Social interaction and the disposition effect. Rev. Financ. Stud. 29, 3177–3209.
Hellwig, M.F., 1980. On the aggregation of information in competitive markets. J. Econom. Theory 22, 477–498.
Hong, H., Kubik, J.D., Stein, J.C., 2004. Social interaction and stock-market participation. J. Finance 59, 137–163.
Hong, H., Kubik, J.D., Stein, J.C., 2005. The neighbor's portfolio: Word-of-mouth effects in the holdings and trades of money managers. J. Finance 60, 2801–2824.
Indjejikian, R., Lu, H., Yang, L., 2014. Rational information leakage. Manage. Sci. 60, 2762–2775.
Jadbabaie, A., Molavi, P., Sandroni, A., Tahbaz-salehi, A., 2012. Non-Bayesian social learning. Games Econom. Behav. 76, 210–225.
Lou, Y. ⓡ Parsa, S. ⓡ Ray, D. ⓡ Li, D. ⓡ Wang, S., 2019. Information aggregation in a financial market with general signal structure. J. Econom. Theory 183, 594–624.
Lou, Y., Wang, S., 2020. A new approach to the existence and regularity of linear equilibrium in a noisy rational expectations economy. J. Math. Econom. 90, 119–126.
Manela, A., 2014. The value of diffusing information. J. Financ. Econ. 111, 181–199.
Molavi, P., Tahbaz-salehi, A., Jadbabaie, A., 2018. A theory of non-Bayesian social learning. Econometrica 86, 445–490.
Ozsoylev, H.N., Walden, J., 2011. Asset pricing in large information networks. J. Econom. Theory 146, 2252–2280.
Ozsoylev, H.N., Walden, J., Yavuz, D., Bildik, R., 2014. Investor networks in the stock market. Rev. Financ. Stud. 27, 1323–1366.
Pool, V.K., Stoffman, N., Yonker, S.E., 2015. The people in your neighborhood: social interactions and mutual fund portfolios. J. Finance 70, 2679–2732.
Rahi, R., Zigrand, J.P., 2018. Information acquisition, price informativeness, and welfare. J. Econ. Theory 177, 558–593.
Verrecchia, R.E., 1982. Information acquisition in a noisy rational expectations economy. Econometrica 50, 1415–1430.
Vives, X., 2008. Information and Learning in Markets: The Impact of Market Microstructure. Princeton University Press, Princeton.
Walden, J., 2019. Trading, profits, and volatility in a dynamic information network model. Rev. Econom. Stud. 86, 2248–2283.
Yang, L., Zhu, H., 2020. Back-running: Seeking and hiding fundamental information in order flows. Rev. Financ. Stud. 33, 1484–1533.