Online Supplemental Material for "The Impact of Relative Wealth Concerns on Wealth Gap and Welfare in a Noisy Rational Expectations Economy"

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In this Supplemental Material we provide the proofs of all propositions in the paper. The numbers for equations correspond to those in the main paper, unless they are specific to the Supplemental Material, in which case the numbers are prefixed by an "S".

Before presenting the proofs, we introduce a lemma that will be useful in our analysis.

Lemma S.1 *Let $X \in \mathbb{R}^n$ be a normally distributed random vector with mean (vector) μ and positive definite covariance matrix Σ, $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix and $q \in \mathbb{R}^n$ is a vector. If $I_n - 2\Sigma A$ is positive definite (where I_n is the identity matrix), then $\mathbb{E}\left[\exp\left(X^\top AX + q^\top X\right)\right]$ is well defined and given by*

$$\begin{aligned}&\mathbb{E}\left[\exp\left(X^\top AX + q^\top X\right)\right]\\&= |I_n - 2\Sigma A|^{-\frac{1}{2}} \exp\left(q^\top \mu + \mu^\top A\mu + \frac{1}{2}(q+2A\mu)^\top (I_n - 2\Sigma A)^{-1}\Sigma(q+2A\mu)\right).\end{aligned}$$

Lemma S.1 provides a formula for the expected value of the exponential function of a quadratic form in a normally distributed random vector. The proof of this lemma can be found in the Appendix of Marín and Rahi (1999) (Lemma A.1) or on page 382 of Vives (2008).

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Proof of Proposition 1.

Here we adopt the proof arguments presented in Proposition 1 in García and Strobl (2011). Let $\bar{x} = \int_0^1 x_i di =: \xi_\lambda + \beta_\lambda\theta - \kappa_\lambda p$ denote the average strategy of all investors in the economy, where $\xi_\lambda := \lambda\xi_H + (1-\lambda)\xi_L$, $\kappa_\lambda := \lambda\kappa_H + (1-\lambda)\kappa_L$ and $\beta_\lambda := \lambda\beta_H + (1-\lambda)\beta_L$.

Let us denote V_i as the relative wealth of investor i, which is given by $V_i = W_i - \gamma\overline{W}$. Consequently, we have:

$$V_i = x_i(\theta - p) - \gamma\bar{x}(\theta - p) = x_i(\theta - p) - \gamma(\xi_\lambda + \beta_\lambda\theta - \kappa_\lambda p)(\theta - p).$$

Upon performing some straightforward calculations, we obtain the following expression:

$$V_i = (x_i + \gamma((\kappa_\lambda - \beta_\lambda)p - \xi_\lambda))(\theta - p) - \gamma\beta_\lambda(\theta - p)^2,$$

which is a quadratic function of the normal random variable $\theta - p$.

Let us introduce some notation to simplify the expressions. We define $\mu_i = \mathbb{E}[\theta - p|y_i, p]$, $\Sigma_i = \mathrm{Var}[\theta - p|y_i, p] = \mathrm{Var}[\theta|y_i, p]$, $A = \rho\gamma\beta_\lambda$ and $q = -\rho(x_i + \gamma((\kappa_\lambda - \beta_\lambda)p - \xi_\lambda))$. With these definitions, we can rewrite the equations as follows:

$$\begin{aligned} 1 - 2\Sigma_i A &= 1 - 2\rho\gamma\beta_\lambda\Sigma_i =: \Psi_i, && \text{(S1)} \\ q^\top\mu_i + \mu_i^\top A\mu_i &= -\rho(x_i + \gamma((\kappa_\lambda - \beta_\lambda)p - \xi_\lambda) - \gamma\beta_\lambda\mu_i)\mu_i =: -\rho(x_i\mu_i + \Upsilon_i), \\ q + 2A\mu_i &= -\rho(x_i + \gamma((\kappa_\lambda - \beta_\lambda)p - \xi_\lambda)) + 2\rho\gamma\beta_\lambda\mu_i =: -\rho\Gamma_i(x_i), \end{aligned}$$

where

$$\Upsilon_i = \gamma((\kappa_\lambda - \beta_\lambda)p - \xi_\lambda - \beta_\lambda\mu_i)\mu_i, \tag{S2}$$

$$\Gamma_i(x_i) = x_i + \gamma((\kappa_\lambda - \beta_\lambda)p - \xi_\lambda - 2\beta_\lambda\mu_i). \tag{S3}$$

With the above identification of A and q, using Lemma S.1,

$$\mathbb{E}[-e^{-\rho V_i}|y_i, p] = -\Psi_i^{-\frac{1}{2}} \exp\left(-\rho\left(\Upsilon_i + x_i\mu_i - \frac{\rho}{2\Psi_i}\Gamma_i(x_i)^2\Sigma_i\right)\right). \tag{S4}$$

Since Ψ_i and Υ_i are independent of x_i, maximizing (S4) with respect to x_i is equivalent to maximizing the expression $x_i\mu_i - \frac{\rho}{2\Psi_i}\Gamma_i(x_i)^2\Sigma_i$. To find the optimal value of x_i, we can take the first-order condition, which gives:

$$\Gamma_i(x_i) = \frac{\mu_i \Psi_i}{\rho\Sigma_i} = \frac{\mu_i}{\rho\Sigma_i} - 2\gamma\beta_\lambda\mu_i. \tag{S5}$$

Therefore,

$$x_i^* = \frac{\mu_i}{\rho\Sigma_i} + \gamma(\xi_\lambda - (\kappa_\lambda - \beta_\lambda)p) = \frac{\mathbb{E}[\theta - p|y_i, p]}{\rho \mathrm{Var}[\theta|y_i, p]} + \gamma(\xi_\lambda - (\kappa_\lambda - \beta_\lambda)p). \tag{S6}$$

Furthermore, since $y_i = \theta + \epsilon_i$ and $p = a + b\theta - sZ$, where $s \neq 0$ (which will be justified later), $\theta, \{\epsilon_i\}_{i\in[0,1]}$ and Z are independent of each other, using projection theorem for normal random variables one can verify that

$$\mathrm{Var}[\theta|y_i, p]^{-1} = \begin{cases} \tau_\theta + \tau_H + \left(\dfrac{b}{s}\right)^2 \tau_z, \; i \in H, \\ \tau_\theta + \tau_L + \left(\dfrac{b}{s}\right)^2 \tau_z, \; i \in L, \end{cases} \tag{S7}$$

and

$$\mathbb{E}[\theta|y_i, p] = \begin{cases} \mu_\theta + \dfrac{\tau_H\left(y_i - \mu_\theta\right) + \frac{b\tau_z}{s^2}(p - \mathbb{E}(p))}{\tau_\theta + \tau_H + \left(\frac{b}{s}\right)^2 \tau_z}, \; i \in H, \\ \mu_\theta + \dfrac{\tau_L\left(y_i - \mu_\theta\right) + \frac{b\tau_z}{s^2}(p - \mathbb{E}(p))}{\tau_\theta + \tau_L + \left(\frac{b}{s}\right)^2 \tau_z}, \; i \in L. \end{cases}$$

Therefore,

$$x_i^* = \begin{cases} \dfrac{\mu_\theta - p}{\rho\mathrm{Var}[\theta|y_i, p]} + \dfrac{\tau_H}{\rho}(y_i - \mu_\theta) + \dfrac{b\tau_z}{\rho s^2}(p - \mathbb{E}(p)) + \gamma\xi_\lambda - \gamma(\kappa_\lambda - \beta_\lambda)p, \; i \in H, \\ \dfrac{\mu_\theta - p}{\rho\mathrm{Var}[\theta|y_i, p]} + \dfrac{\tau_L}{\rho}(y_i - \mu_\theta) + \dfrac{b\tau_z}{\rho s^2}(p - \mathbb{E}(p)) + \gamma\xi_\lambda - \gamma(\kappa_\lambda - \beta_\lambda)p, \; i \in L. \end{cases} \tag{S8}$$

Matching coefficients in (S8) and the conjectured trading strategies leads to the following expressions:

$$\beta_H = \frac{\tau_H}{\rho}, \tag{S9}$$

$$\beta_L = \frac{\tau_L}{\rho}, \tag{S10}$$

$$\kappa_H = \frac{\tau_\theta + \tau_H + (\frac{b}{s})^2\tau_z}{\rho} - \frac{b\tau_z}{\rho s^2} + \gamma(\kappa_\lambda - \beta_\lambda), \tag{S11}$$

$$\kappa_L = \frac{\tau_\theta + \tau_L + (\frac{b}{s})^2\tau_z}{\rho} - \frac{b\tau_z}{\rho s^2} + \gamma(\kappa_\lambda - \beta_\lambda), \tag{S12}$$

$$\xi_L = \xi_H = \frac{\tau_\theta + (\frac{b}{s})^2\tau_z}{\rho}\mu_\theta - \frac{b\tau_z}{\rho s^2}\mathbb{E}(p) + \gamma\xi_\lambda. \tag{S13}$$

Furthermore, the market clearing condition can be alternatively written as

$$\int_0^1 x_i^* di = (1-\lambda)(\xi_L + \beta_L\theta - \kappa_L p) + \lambda(\xi_H + \beta_H\theta - \kappa_H p) = Z = \frac{1}{s}(a + b\theta - p).$$

Matching coefficients leads to the following equations:

$$\frac{b}{s} = \beta_\lambda = \lambda\beta_H + (1-\lambda)\beta_L, \tag{S14}$$

$$\frac{1}{s} = \kappa_\lambda = \lambda\kappa_H + (1-\lambda)\kappa_L, \tag{S15}$$

$$\frac{a}{s} = \xi_\lambda = \lambda\xi_H + (1-\lambda)\xi_L. \tag{S16}$$

To show the existence and uniqueness of linear equilibria, it is necessary and sufficient to show that the system of equations (S9)-(S16) has a unique solution $(a, b, s, \xi_H, \beta_H, \kappa_H, \xi_L, \beta_L, \kappa_L)$. Firstly, from (S14) and (S15), we know that (S11) and (S12) can be respectively translated to:

$$\kappa_H = \frac{\tau_\theta + \tau_H + \beta_\lambda^2\tau_z}{\rho} - \frac{\tau_z\beta_\lambda}{\rho}\kappa_\lambda + \gamma\kappa_\lambda - \gamma\beta_\lambda, \tag{S17}$$

$$\kappa_L = \frac{\tau_\theta + \tau_L + \beta_\lambda^2\tau_z}{\rho} - \frac{\tau_z\beta_\lambda}{\rho}\kappa_\lambda + \gamma\kappa_\lambda - \gamma\beta_\lambda. \tag{S18}$$

As a result,

$$\kappa_\lambda = \lambda\kappa_H + (1-\lambda)\kappa_L = \frac{\tau_\theta + \beta_\lambda^2\tau_z}{\rho} + \frac{\lambda\tau_H + (1-\lambda)\tau_L}{\rho} - \frac{\tau_z\beta_\lambda}{\rho}\kappa_\lambda + \gamma\kappa_\lambda - \gamma\beta_\lambda,$$

which is equivalent to

$$\left(1-\gamma+\frac{\beta_\lambda\tau_z}{\rho}\right)\kappa_\lambda=\frac{\tau_\theta+\beta_\lambda^2\tau_z}{\rho}+(1-\gamma)\beta_\lambda.$$

Consequently,

$$\kappa_\lambda=\frac{\tau_\theta+\beta_\lambda^2\tau_z+\rho(1-\gamma)\beta_\lambda}{\rho(1-\gamma)+\beta_\lambda\tau_z}=\frac{\tau_\theta}{\rho(1-\gamma)+\beta_\lambda\tau_z}+\beta_\lambda. \tag{S19}$$

Substitute (S19) into (S17) and (S18) we get

$$\begin{aligned}\kappa_H=&\frac{\tau_\theta+\tau_H+\beta_\lambda^2\tau_z}{\rho}-\frac{\tau_z\beta_\lambda}{\rho}\cdot\frac{\tau_\theta}{\rho(1-\gamma)+\beta_\lambda\tau_z}-\frac{\beta_\lambda^2\tau_z}{\rho}+\frac{\gamma\tau_\theta}{\rho(1-\gamma)+\beta_\lambda\tau_z}\\=&\frac{\tau_\theta+\tau_H}{\rho}-\frac{\tau_\theta}{\rho}\cdot\frac{\beta_\lambda\tau_z}{\rho(1-\gamma)+\beta_\lambda\tau_z}+\frac{\tau_\theta}{\rho}\cdot\frac{\rho\gamma}{\rho(1-\gamma)+\beta_\lambda\tau_z}\\=&\frac{\tau_\theta}{\rho}\left(1-\frac{\beta_\lambda\tau_z}{\rho(1-\gamma)+\beta_\lambda\tau_z}+\frac{\rho\gamma}{\rho(1-\gamma)+\beta_\lambda\tau_z}\right)+\frac{\tau_H}{\rho}\\=&\frac{\tau_\theta}{\rho(1-\gamma)+\beta_\lambda\tau_z}+\frac{\tau_H}{\rho},\end{aligned}$$

and

$$\kappa_L=\frac{\tau_\theta}{\rho(1-\gamma)+\beta_\lambda\tau_z}+\frac{\tau_L}{\rho}.$$

Substitute (S14)-(S16) into (S13) we have

$$\begin{aligned}\xi_L=\xi_H=&\frac{\tau_\theta+\beta_\lambda^2\tau_z}{\rho}\mu_\theta-\frac{\beta_\lambda\tau_z}{\rho}(\xi_\lambda+\beta_\lambda\mu_\theta)+\gamma\xi_\lambda\\=&\frac{\tau_\theta\mu_\theta}{\rho(1-\gamma)+\beta_\lambda\tau_z}.\end{aligned}$$

From (S14)-(S16) we know that

$$\begin{aligned}a&=\frac{\xi_\lambda}{\kappa_\lambda}=\frac{\tau_\theta\mu_\theta}{\tau_\theta+\beta_\lambda\rho(1-\gamma)+\beta_\lambda^2\tau_z}.\\b&=\frac{\beta_\lambda}{\kappa_\lambda}=1-\frac{\tau_\theta}{\tau_\theta+\beta_\lambda\rho(1-\gamma)+\beta_\lambda^2\tau_z},\\s&=\frac{1}{\kappa_\lambda}=\frac{\rho(1-\gamma)+\beta_\lambda\tau_z}{\tau_\theta+\beta_\lambda\rho(1-\gamma)+\beta_\lambda^2\tau_z},\end{aligned}$$

where $s \neq 0$ under the condition $\gamma \neq 1 + \frac{\beta_\lambda \tau_z}{\rho}$ in the proposition.

Finally, according to (S1), we see that the second-order condition $\Psi_i = 1 - 2\rho\gamma\beta_\lambda\Sigma_i > 0$ is the same as the required condition when using Lemma S.1 and it holds under the condition $\gamma < \frac{\tau_\theta + \tau_L + \beta_\lambda^2 \tau_z}{2\rho\beta_\lambda}$ in the proposition. This completes the proof. □

Proof of Proposition 2.

In the proof, we denote $\alpha = \frac{\tau_\theta + \tau_L + \beta_\lambda^2 \tau_z}{2\rho\beta_\lambda}$ for simplicity. Utilizing Proposition 1, we can express the average wealth gap between L and H as follows:

$$\mathbb{E}(W_H - W_L) = \mathbb{E}[(x_H - x_L)(\theta - p)] = \frac{\tau_H - \tau_L}{\rho} \mathbb{E}[(\theta - p)^2]. \tag{S20}$$

Since $\theta - p$ follows a normal distribution, with mean

$$\mathbb{E}(\theta - p) = \frac{-\tau_\theta \mu_\theta + \tau_\theta \mu_\theta}{\tau_\theta + \beta_\lambda \rho (1-\gamma) + \beta_\lambda^2 \tau_z} = 0, \tag{S21}$$

and variance

$$\mathrm{Var}(\theta - p) = \frac{\tau_\theta + (\rho(1-\gamma) + \beta_\lambda \tau_z)^2 \tau_z^{-1}}{(\tau_\theta + \beta_\lambda \rho(1-\gamma) + \beta_\lambda^2 \tau_z)^2}, \tag{S22}$$

we have

$$\begin{aligned} \mathbb{E}[(\theta - p)^2] &= \mathrm{Var}(\theta - p) + (\mathbb{E}(\theta - p))^2 \\ &= \mathrm{Var}(\theta - p). \end{aligned} \tag{S23}$$

Therefore,

$$\begin{aligned} \frac{\partial \mathbb{E}[(\theta - p)^2]}{\partial \gamma} = \frac{\partial \mathrm{Var}(\theta - p)}{\partial \gamma} &= \frac{[\tau_\theta + \beta_\lambda \rho(1-\gamma) + \beta_\lambda^2 \tau_z][-2\rho^2(1-\gamma)\tau_z^{-1} - 2\beta_\lambda \rho]}{(\tau_\theta + \beta_\lambda \rho(1-\gamma) + \beta_\lambda^2 \tau_z)^3} \\ &\quad + \frac{2\rho\beta_\lambda[\tau_\theta + (\rho(1-\gamma) + \beta_\lambda \tau_z)^2 \tau_z^{-1}]}{(\tau_\theta + \beta_\lambda \rho(1-\gamma) + \beta_\lambda^2 \tau_z)^3} \\ &= -\frac{2\rho^2(1-\gamma)\tau_\theta \tau_z^{-1}}{(\tau_\theta + \beta_\lambda \rho(1-\gamma) + \beta_\lambda^2 \tau_z)^3}, \end{aligned} \tag{S24}$$

which implies that the average wealth gap between L and H is decreasing in $\gamma \in (0, \alpha)$ if $\alpha < 1$, and firstly decreasing in $\gamma \in (0, 1)$ and eventually increasing in $(1, \alpha)$ if $\alpha > 1$. Note that by the relation

$$\frac{\tau_\theta + \tau_L + \beta_\lambda^2 \tau_z}{2\rho\beta_\lambda} \leq \frac{\tau_\theta + \rho\beta_\lambda + \beta_\lambda^2 \tau_z}{2\rho\beta_\lambda} < 1 + \frac{\tau_\theta + \beta_\lambda^2 \tau_z}{\rho\beta_\lambda},$$

we know that under the condition $\gamma < \frac{\tau_\theta+\tau_L+\beta_\lambda^2\tau_z}{2\rho\beta_\lambda}$, the denominator in (S24) is positive. This completes the proof of the monotonicity of the average wealth gap in Parts (i) and (ii).

We now show the monotonicity of investors' welfare. Substitute (S5) and (S6) into (S4) and note (S1)-(S3), we have

$$\begin{aligned}
&\Upsilon_i + x_i\mu_i - \frac{\rho}{2\Psi_i}\Gamma_i(x_i)^2\Sigma_i \\
&= \gamma((\kappa_\lambda - \beta_\lambda)p - \xi_\lambda - \beta_\lambda\mu_i)\mu_i + \frac{\mu_i^2}{\rho\Sigma_i} + \gamma(\xi_\lambda - (\kappa_\lambda - \beta_\lambda)p)\mu_i - \frac{\rho\Sigma_i}{2\Psi_i}\left(\frac{\mu_i}{\rho\Sigma_i} - 2\gamma\beta_\lambda\mu_i\right)^2 \\
&= -\gamma\beta_\lambda\mu_i^2 + \frac{\mu_i^2}{\rho\Sigma_i} - \frac{\mu_i^2}{2\Psi_i\rho\Sigma_i} + \frac{2\gamma\beta_\lambda\mu_i^2}{\Psi_i} - \frac{2\rho\Sigma_i\gamma^2\beta_\lambda^2\mu_i^2}{\Psi_i} \\
&= -\frac{\gamma\beta_\lambda\mu_i^2 - 2\rho\Sigma_i\gamma^2\beta_\lambda^2\mu_i^2}{\Psi_i} + \frac{\mu_i^2 - 2\rho\beta_\lambda\gamma\Sigma_i\mu_i^2}{\Psi_i\rho\Sigma_i} - \frac{\mu_i^2}{2\Psi_i\rho\Sigma_i} + \frac{2\gamma\beta_\lambda\mu_i^2}{\Psi_i} - \frac{2\rho\Sigma_i\gamma^2\beta_\lambda^2\mu_i^2}{\Psi_i} \\
&= \frac{\gamma\beta_\lambda\mu_i^2}{\Psi_i} - \frac{2\rho\beta_\lambda\gamma\Sigma_i\mu_i^2}{\Psi_i\rho\Sigma_i} + \frac{\mu_i^2}{2\Psi_i\rho\Sigma_i} \\
&= \frac{\mu_i^2}{2\Psi_i\rho\Sigma_i} - \frac{\rho\beta_\lambda\gamma\Sigma_i\mu_i^2}{\Psi_i\rho\Sigma_i} \\
&= \frac{\mu_i^2}{2\rho\Sigma_i}.
\end{aligned}$$

Therefore, investor i's maximum utility conditional on her signal y_i and price p can be written as

$$\mathbb{E}[u(V_i)|y_i, p] = -\Psi_i^{-\frac{1}{2}} \exp\left(-\frac{\mu_i^2}{2\Sigma_i}\right),$$

where $\Sigma_i^{-1} = \tau_\theta + \tau_i + \beta_\lambda^2\tau_z$, and $\mu_i = \mathbb{E}[\theta - p|y_i, p]$ is a normally distributed random variable. Since μ_i follows a normal distribution and $A = -\frac{1}{2\Sigma_i} < 0$, which implies $1 - 2\text{Var}(\mu_i)A$ is positive, we can calculate that

$$\begin{aligned}
\mathbb{E}[u(V_i)] &= \mathbb{E}(\mathbb{E}[u(V_i)|y_i, p]) \\
&= -\Psi_i^{-\frac{1}{2}}\left(1 + \frac{\text{Var}(\mu_i)}{\Sigma_i}\right)^{-\frac{1}{2}} \exp\left(-\frac{[\mathbb{E}(\mu_i)]^2}{2\Sigma_i}\left(1 - \frac{\text{Var}(\mu_i)}{\Sigma_i + \text{Var}(\mu_i)}\right)\right)
\end{aligned}$$

$$= -\left(\Psi_i \frac{\mathrm{Var}(\theta - p)}{\Sigma_i}\right)^{-\frac{1}{2}} \exp\left(-\frac{(\mu_\theta - \mathbb{E}(p))^2}{2\mathrm{Var}(\theta - p)}\right)$$

$$= -\left(\frac{\Psi_i}{\Sigma_i}\mathrm{Var}(\theta - p)\right)^{-\frac{1}{2}}, \tag{S25}$$

where the second equality uses Lemma S.1, the third one uses the law of total variance: $\mathrm{Var}(X) = \mathrm{Var}(\mathbb{E}[X|\mathcal{F}]) + \mathbb{E}(\mathrm{Var}[X|\mathcal{F}])$, and the last one follows from (S21). According to (S25), we next analyze the impact of γ on $\frac{\Psi_i}{\Sigma_i}\mathrm{Var}(\theta - p)$ instead of $\mathbb{E}[u(V_i)]$ since they are monotonically equivalent.

Based on equations (S7) and (S14), we observe that $\Sigma_i^{-1} = \mathrm{Var}[\theta|y_i, p]^{-1} = \tau_\theta + \tau_i + \beta_\lambda^2\tau_z$, where $\tau_i = \tau_H$ for $i \in H$ and $\tau_j = \tau_L$ for $j \in L$. This implies that Σ_i^{-1} does not depend on γ. Consequently, we have $\Psi_i/\Sigma_i = \Sigma_i^{-1} - 2\rho\gamma\beta_\lambda = \tau_\theta + \tau_i + \beta_\lambda^2\tau_z - 2\rho\gamma\beta_\lambda$. It is evident that Ψ_i/Σ_i is a strictly decreasing function of γ. Considering equation (S24), we can conclude that $\mathrm{Var}(\theta - p)$ and, consequently, $\frac{\Psi_i}{\Sigma_i}\mathrm{Var}(\theta - p)$ strictly decrease as γ varies within the interval $(0, \alpha)$, under the condition $\alpha \leq 1$.

We proceed to analyze the case when $\alpha > 1$. Following the previous analysis, we can similarly demonstrate that $\frac{\Psi_i}{\Sigma_i}\mathrm{Var}(\theta - p)$ is strictly decreasing in $\gamma \in (0, 1)$. Recalling (S22), we have

$$\frac{\Psi_i}{\Sigma_i}\mathrm{Var}(\theta - p) = (\Sigma_i^{-1} - 2\rho\gamma\beta_\lambda)\frac{\tau_\theta + (\rho(1-\gamma) + \beta_\lambda\tau_z)^2\tau_z^{-1}}{(\tau_\theta + \beta_\lambda\rho(1-\gamma) + \beta_\lambda^2\tau_z)^2}.$$

Therefore,

$$\begin{aligned}\frac{\partial(\Psi_i\mathrm{Var}(\theta - p)/\Sigma_i)}{\partial\gamma} =& \frac{-2\rho\beta_\lambda(\tau_\theta + (\rho(1-\gamma) + \beta_\lambda\tau_z)^2\tau_z^{-1})(\tau_\theta + \beta_\lambda\rho(1-\gamma) + \beta_\lambda^2\tau_z)}{(\tau_\theta + \beta_\lambda\rho(1-\gamma) + \beta_\lambda^2\tau_z)^3} \\ &- \frac{(\Sigma_i^{-1} - 2\rho\gamma\beta_\lambda)(2\rho\beta_\lambda + 2\rho^2(1-\gamma)\tau_z^{-1})(\tau_\theta + \beta_\lambda\rho(1-\gamma) + \beta_\lambda^2\tau_z)}{(\tau_\theta + \beta_\lambda\rho(1-\gamma) + \beta_\lambda^2\tau_z)^3} \\ &+ \frac{2\rho\beta_\lambda(\Sigma_i^{-1} - 2\rho\gamma\beta_\lambda)(\tau_\theta + (\rho(1-\gamma) + \beta_\lambda\tau_z)^2\tau_z^{-1})}{(\tau_\theta + \beta_\lambda\rho(1-\gamma) + \beta_\lambda^2\tau_z)^3}\end{aligned} \tag{S26}$$

The denominator of the three terms at the right-hand side of (S26) is positive and the sum of the three numerators, denoted as $h_i(\gamma)$, is a cubic function of γ:

$$h_i(\gamma) = 2\rho^4\beta_\lambda^2\tau_z^{-1}\gamma^3 - 6(\rho^4\beta_\lambda^2\tau_z^{-1} + \rho^3\beta_\lambda^3 + \rho^3\beta_\lambda\tau_\theta\tau_z^{-1})\gamma^2$$

$$+ 2[3\rho^4\beta_\lambda^2\tau_z^{-1} + 6\rho^3\beta_\lambda^3 + 4\rho^3\beta_\lambda\tau_\theta\tau_z^{-1} + 3\rho^2\beta_\lambda^2(\tau_\theta + \beta_\lambda^2\tau_z) + \rho^2\tau_\theta\tau_z^{-1}\Sigma_i^{-1}]\gamma$$
$$- (2\rho^4\beta_\lambda^2\tau_z^{-1} + 6\rho^3\beta_\lambda^3 + 2\rho^3\beta_\lambda\tau_\theta\tau_z^{-1} + 6\rho^2\beta_\lambda^2(\tau_\theta + \beta_\lambda^2\tau_z) + 2\rho^2\tau_\theta\tau_z^{-1}\Sigma_i^{-1} + 2\rho\beta_\lambda(\tau_\theta + \beta_\lambda^2\tau_z)^2).$$

It is clear that $h_i(0) < 0$, $h_i'(0) > 0$. Some simple calculations lead to

$$h_i''(\gamma) \propto \rho^4\beta_\lambda^2\tau_z^{-1}\gamma - \rho^4\beta_\lambda^2\tau_z^{-1} - \rho^3\beta_\lambda^3 - \rho^3\beta_\lambda\tau_\theta\tau_z^{-1},$$
$$h_i''\left(1 + \frac{\tau_\theta + \beta_\lambda^2\tau_z}{\rho\beta_\lambda}\right) \propto \rho^4\beta_\lambda^2\tau_z^{-1} + \rho^3\beta_\lambda\tau_z^{-1}(\tau_\theta + \beta_\lambda^2\tau_z) - \rho^4\beta_\lambda^2\tau_z^{-1} - \rho^3\beta_\lambda^3 - \rho^3\beta_\lambda\tau_\theta\tau_z^{-1} = 0.$$

Based on the previous two relations and the inequality $\alpha = \frac{\tau_\theta+\tau_L+\beta_\lambda^2\tau_z}{2\rho\beta_\lambda} \leq 1 + \frac{\tau_\theta+\beta_\lambda^2\tau_z}{\rho\beta_\lambda}$, it follows that $h_i''(\gamma) < 0$ for $\gamma \in (0, \alpha)$. Therefore, the function $h_i(\cdot)$ is strictly concave over the interval $(0, \alpha)$, implying three possibilities for the monotonicity of investor welfare:

(i) investor welfare strictly decreases in $\gamma \in (0, \alpha)$. This occurs when $\max_{0<\gamma<\alpha} h_i(\gamma) \leq 0$.

(ii) there exist $1 < \gamma_i^* < \alpha$ such that the welfare of investors first strictly decreases on $(0, \gamma_i^*)$, then strictly increases on (γ_i^*, α). This case happens when $\max_{0<\gamma<\alpha} h_i(\gamma) > 0$ and $h_i(\alpha) \geq 0$.

(iii) there exist $1 < \gamma_i^* < \hat{\gamma}_i < \alpha$ such that the welfare of investors first strictly decreases on $(0, \gamma_i^*)$, then strictly increases on $(\gamma_i^*, \hat{\gamma}_i)$, and eventually strictly decreases on $(\hat{\gamma}_i, \alpha)$. This case happens when $\max_{0<\gamma<\alpha} h_i(\gamma) > 0$ and $h_i(\alpha) < 0$.

In the second and third cases, the fact that $\gamma_i^* > 1$ is due to the observation that $\frac{\Psi_i}{\Sigma_i}\mathrm{Var}(\theta - p)$ is strictly decreasing in $\gamma \in (0, 1)$. Furthermore, it is evident that γ_i^* can only take two values depending on whether $i \in H$ or $i \in L$.

Finally, at the end of the proof, we assert that the second case is impossible for the low-precision investor $i \in L$. This is because at the endpoint $\gamma = \alpha = \frac{\tau_\theta+\tau_L+\beta_\lambda^2\tau_z}{2\rho\beta_\lambda}$, it holds that

$$\frac{\Psi_i}{\Sigma_i}\mathrm{Var}(\theta - p) = (\tau_\theta + \tau_i + \beta_\lambda^2\tau_z - 2\rho\gamma\beta_\lambda)\mathrm{Var}(\theta - p) = 0$$

for $i \in L$. This completes the proof. □

Proof of Proposition 3.

We now demonstrate that the solution β_λ^{**} does not correspond to a stable equilibrium. Let $\delta > 0$ be small enough. According to (6) and (7), for $\beta_{\lambda-} := \beta_\lambda^{**} - \delta$, we have

$$\begin{aligned}
&\tau_z \beta_{\lambda-}^2 - 2\rho\gamma\beta_{\lambda-} - \widehat{C} > 0, \\
\Longleftrightarrow\ &\tau_\theta + \tau_H + \beta_{\lambda-}^2 \tau_z - 2\rho\gamma\beta_{\lambda-} < e^{2\rho(c_1-c_2)} \left(\tau_\theta + \tau_L + \beta_{\lambda-}^2 \tau_z - 2\rho\gamma\beta_{\lambda-}\right),
\end{aligned}$$

which implies that under the fraction $\lambda = \frac{\rho\beta_{\lambda-} - \tau_L}{\tau_H - \tau_L}$, investor $j \in L$ can generate more profit than investor $i \in H$. Thus, the willingness of investors in this market to buy high-precision signals decreases, leading to a greater deviation from equilibrium.

Similarly, let $\beta_{\lambda+} = \beta_\lambda^{**} + \delta$, then

$$\begin{aligned}
&\tau_z \beta_{\lambda+}^2 - 2\rho\gamma\beta_{\lambda+} - \widehat{C} < 0, \\
\Longleftrightarrow\ &\tau_\theta + \tau_H + \beta_{\lambda+}^2 \tau_z - 2\rho\gamma\beta_{\lambda+} > e^{2\rho(c_1-c_2)} \left(\tau_\theta + \tau_L + \beta_{\lambda+}^2 \tau_z - 2\rho\gamma\beta_{\lambda+}\right),
\end{aligned}$$

which shows that under the fraction $\lambda = \frac{\rho\beta_{\lambda+} - \tau_L}{\tau_H - \tau_L}$, each investor $i \in H$ can generate more profit than investor $j \in L$. Thus, the willingness of investors in this market to buy high-precision signals increases, which will also lead to a greater deviation from equilibrium. In conclusion, β_λ^{**} would never lead to a stable equilibrium. Similarly, we can demonstrate that the fraction $\frac{\rho\beta_\lambda^* - \tau_L}{\tau_H - \tau_L}$ is stable using the above arguments. Moreover, it is an interior equilibrium when $\beta_\lambda^* \in \left(\frac{\tau_L}{\rho}, \frac{\tau_H}{\rho}\right)$.

Furthermore, when $\beta_\lambda^* > \tau_L/\rho > 0$, by the definitions of β_λ^* and $\widehat{C}$, we have

$$\begin{aligned}
\tau_\theta + \tau_L + (\beta_\lambda^*)^2 \tau_z - 2\rho\beta_\lambda^*\gamma &= \tau_\theta + \tau_L + \widehat{C} \\
&= \tau_\theta + \tau_L + \frac{\tau_\theta + \tau_H - e^{2\rho(c_1-c_2)}(\tau_\theta + \tau_L)}{e^{2\rho(c_1-c_2)} - 1} \\
&= \frac{\tau_H - \tau_L}{e^{2\rho(c_1-c_2)} - 1} \\
&> 0.
\end{aligned}$$

Thus, the condition $\gamma < \frac{\tau_\theta + \tau_L + (\beta_\lambda^*)^2 \tau_z}{2\rho\beta_\lambda^*}$ required for the equilibrium existence in Proposition 1 naturally holds under the condition of $\beta_\lambda^* > \tau_L/\rho > 0$. Moreover, the condition

$$\gamma \neq 1 + \beta_\lambda \tau_z \rho^{-1} = 1 + \gamma + \sqrt{\gamma^2 + \rho^{-2}\widehat{C}\tau_z}$$

in Proposition 1 also naturally holds in the case of endogenous information. This completes the proof. □

Proof of Proposition 4.

Under the conditions specified in Proposition 3, the value β_λ^* can exclusively result in a stable interior equilibrium. For the sake of simplicity, we will use the notation β_λ instead of β_λ^* throughout the proof. Thus,

$$\beta_\lambda = \frac{1}{\tau_z}\left(\rho\gamma + \sqrt{\rho^2\gamma^2 + \widehat{C}\tau_z}\right), \tag{S27}$$

and

$$\frac{\partial \beta_\lambda}{\partial \gamma} = \frac{\rho}{\tau_z} + \frac{\rho^2\gamma}{\tau_z\sqrt{\rho^2\gamma^2 + \widehat{C}\tau_z}} > 0. \tag{S28}$$

Recall the expression in (S25) that

$$\mathbb{E}[u(V_i)] = -\left(\frac{\Psi_i}{\Sigma_i}\mathrm{Var}(\theta - p)\right)^{-\frac{1}{2}}.$$

We know that $\Psi_i/\Sigma_i = \Sigma_i^{-1} - 2\rho\gamma\beta_\lambda$, where $\Sigma_i^{-1} = \tau_\theta + \tau_H + \beta_\lambda^2\tau_z$ for $i \in [0,\lambda]$ and $\Sigma_i^{-1} = \tau_\theta + \tau_L + \beta_\lambda^2\tau_z$ for $i \in (\lambda, 1]$. Thus

$$\begin{aligned}
\frac{\partial(\Psi_i/\Sigma_i)}{\partial\gamma} &= 2\beta_\lambda\tau_z\frac{\partial\beta_\lambda}{\partial\gamma} - 2\rho\gamma\frac{\partial\beta_\lambda}{\partial\gamma} - 2\rho\beta_\lambda \\
&= 2\frac{\partial\beta_\lambda}{\partial\gamma}\sqrt{\rho^2\gamma^2 + \widehat{C}\tau_z} - 2\rho\beta_\lambda \\
&= 0,
\end{aligned}$$

where the second equality follows from (S27) and the last one from (S27) and (S28). Therefore,

$$\frac{\partial \mathbb{E}[u(V_i)]}{\partial \gamma} \propto \frac{\partial \mathrm{Var}(\theta - p)}{\partial \gamma}.$$

By (S22) and (S28), we have

$$\begin{aligned}\frac{\partial \mathrm{Var}(\theta - p)}{\partial \gamma} &= -\frac{2\rho^2(1-\gamma)\tau_\theta \tau_z^{-1}}{(\tau_\theta + \beta_\lambda \rho(1-\gamma) + \beta_\lambda^2 \tau_z)^3} \\ &\quad - \frac{2\frac{\partial \beta_\lambda}{\partial \gamma}[\beta_\lambda \tau_z \tau_\theta + \beta_\lambda^3 \tau_z^2 + \rho^3(1-\gamma)^3 \tau_z^{-1} + 3\rho^2(1-\gamma)^2 \beta_\lambda + 3\rho(1-\gamma)\beta_\lambda^2 \tau_z]}{(\tau_\theta + \beta_\lambda \rho(1-\gamma) + \beta_\lambda^2 \tau_z)^3}. \quad \text{(S29)}\end{aligned}$$

Since the denominator of the two terms in (S29) is positive, to show the conclusion it suffices to show that

$$\rho^2(1-\gamma)\tau_\theta \tau_z^{-1} + \frac{\partial \beta_\lambda}{\partial \gamma}[\beta_\lambda \tau_z \tau_\theta + \beta_\lambda^3 \tau_z^2 + \rho^3(1-\gamma)^3 \tau_z^{-1} + 3\rho^2(1-\gamma)^2 \beta_\lambda + 3\rho(1-\gamma)\beta_\lambda^2 \tau_z] > 0.$$

Substitute the expressions of β_λ and $\frac{\partial \beta_\lambda}{\partial \gamma}$ in (S27) and (S28) into the left-hand side of the above relation, we have

$$\begin{aligned}&\rho^2(1-\gamma)\tau_\theta \tau_z^{-1} + \frac{\partial \beta_\lambda}{\partial \gamma}[\beta_\lambda \tau_z \tau_\theta + \beta_\lambda^3 \tau_z^2 + \rho^3(1-\gamma)^3 \tau_z^{-1} + 3\rho^2(1-\gamma)^2 \beta_\lambda + 3\rho(1-\gamma)\beta_\lambda^2 \tau_z] \\ &= \rho^2(1-\gamma)\tau_\theta \tau_z^{-1} \\ &+ \left(\frac{\rho}{\tau_z} + \frac{\rho^2\gamma}{\tau_z\sqrt{\rho^2\gamma^2 + \widehat{C}\tau_z}}\right)[\beta_\lambda \tau_z \tau_\theta + \beta_\lambda^3 \tau_z^2 + \rho^3(1-\gamma)^3 \tau_z^{-1} + 3\rho^2(1-\gamma)^2 \beta_\lambda + 3\rho(1-\gamma)\beta_\lambda^2 \tau_z] \\ &\propto \rho^2(1-\gamma)\tau_\theta + \left(\rho + \frac{\rho^2\gamma}{\sqrt{\rho^2\gamma^2 + \widehat{C}\tau_z}}\right)\left(\rho\gamma + \sqrt{\rho^2\gamma^2 + \widehat{C}\tau_z}\right)\tau_\theta \\ &+ \left(\rho + \frac{\rho^2\gamma}{\sqrt{\rho^2\gamma^2 + \widehat{C}\tau_z}}\right)[\beta_\lambda^3 \tau_z^2 + \rho^3(1-\gamma)^3 \tau_z^{-1} + 3\rho^2(1-\gamma)^2 \beta_\lambda + 3\rho(1-\gamma)\beta_\lambda^2 \tau_z] \\ &\propto \rho^2\tau_\theta + \left(\rho + \frac{\rho^2\gamma}{\sqrt{\rho^2\gamma^2 + \widehat{C}\tau_z}}\right)\sqrt{\rho^2\gamma^2 + \widehat{C}\tau_z}\tau_\theta + \frac{\rho^3\gamma^2}{\sqrt{\rho^2\gamma^2 + \widehat{C}\tau_z}}\tau_\theta \\ &+ \left(\rho + \frac{\rho^2\gamma}{\sqrt{\rho^2\gamma^2 + \widehat{C}\tau_z}}\right)[\beta_\lambda^3 \tau_z^2 + \rho^3(1-\gamma)^3 \tau_z^{-1} + 3\rho^2(1-\gamma)^2 \beta_\lambda + 3\rho(1-\gamma)\beta_\lambda^2 \tau_z].\end{aligned}$$

Therefore, it further suffices to show that $\beta_\lambda^3\tau_z^2+\rho^3(1-\gamma)^3\tau_z^{-1}+3\rho^2(1-\gamma)^2\beta_\lambda+3\rho(1-\gamma)\beta_\lambda^2\tau_z > 0$. In fact,

$$\begin{aligned}
\beta_\lambda^3\tau_z^2 + \rho^3(1-\gamma)^3\tau_z^{-1} + 3\rho^2(1-\gamma)^2\beta_\lambda + 3\rho(1-\gamma)\beta_\lambda^2\tau_z &= \tau_z^{-1}(\beta_\lambda\tau_z + \rho(1-\gamma))^3 \\
&= \tau_z^{-1}\left(\rho\gamma + \sqrt{\rho^2\gamma^2 + \widehat{C}\tau_z} + \rho(1-\gamma)\right)^3 \\
&= \tau_z^{-1}\left(\rho + \sqrt{\rho^2\gamma^2 + \widehat{C}\tau_z}\right)^3 \\
&> 0,
\end{aligned}$$

where the second equality follows from (S27). This completes the proof. □

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