How many financial advisers do you need?

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Abstract

We study a rational expectations equilibrium economy populated by investors and their financial advisers. Lacking financial literacy, investors depend on advisers for investment recommendations. Advisers provide suggestions that are in the best interest of their client investors, which then aggregate suggested strategies under bounded rationality constraints of conformism and robust regret aversion. Our model can explain why investors require financial advice and why they consult only a few advisers despite the abundance of available options. Our analysis highlights that quality of advice is more important than the number of advisers consulted, and that it is never optimal to rely on a single financial adviser.

Keywords: Rational Expectations Equilibrium, Investment Advice, Optimal Aggregation, Information Acquisition

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1 Introduction

Many investors turn to financial advisers to form their investment strategies. However, despite the wide array of advisers available, investors rarely consult all possible sources. This behavior is inconsistent with traditional models of fully rational agents in rational expectations equilibrium economies, where investors would optimally incorporate all accessible information. Selective consultation with a small number of financial advisers points to underlying cognitive or resource constraints of investors. How many financial advisers do investors of bounded rationality need? And how much to spend on each adviser?

The goal of this paper is to formally study these questions within a rational expectations equilibrium framework. We extend the classical model of Hellwig (1980) by introducing two types of agents: Investors of bounded rationality and their financial advisers. Unlike fully rational agents, these investors lack full financial literacy and do not know or understand the key market parameters needed to construct optimal investment strategies. Instead, they depend on financial advisers who observe private signals about the fundamentals of the risky asset and recommend strategies maximizing their client investor's expected utility.

Upon receiving suggested investment strategies from their advisers, investors optimally aggregate these suggestions under two further constraints of bounded rationality: Conformism and robust regret aversion. Conformism reflects a behavioral bias where investors align their actions with the direction of the advice. They do not short the stock if all advisers recommend a long position, and vice versa. Robust regret aversion requires that the aggregated strategy must outperform any single adviser's strategy. Because the investor does not know market parameters, we require it to hold for any value thereof.

We herein consider two basic setups: One where quality of advisers in terms of signal precision is exogenous and another where it is endogenous and controlled by the investors. The first setup applies for investors that, for example, read about a suggested investment strategy in the financial press. In this setting, it is plausible that the investor has knowledge about the precision of the advisers inferred from their reputation and past performance. But the investor cannot influence this precision. In the second setup, investors can affect the precision of their advisers at a cost. The larger the expense on a given adviser, the more precise the signal based on which the adviser suggests an investment strategy.

In the first setup with exogenous quality of advisers, investors can observe suggested strategies of up to n financial advisers, and doing so is free of charge. We suppose that each investor knows the precision of signals based upon which advisers suggest investment strategies. With how many advisers should the investor consult in this setting? And how to optimally aggregate suggested strategies? We are able to answer both questions analytically and gain the following insights. First, the investor should always consult with at least two advisers, no matter how large the difference in the precision of their signals. This result is remarkable since all advisers in our model suggest investment strategies that are in the best interest of the investor and there are no safety or fraud concerns. Still, it is never optimal to only consult with a single adviser. Second, investors should not always consult with all available advisers. When the precision of an additional adviser is low relative to the typical precision of existing advisers it is best to ignore the suggestions of the additional adviser. Third, when the number of potential advisers is large, a given investor should consult with all of them if and only if they are relatively homogenous in terms of signal precision. In the case that the signal precision of advisers is completely identical it is optimal to consult with all of them and equal-weight their suggestions.

Our model can simultaneously explain the need for financial advice and why investors consult only a small number of advisers. Due to limited financial literacy, investors cannot infer and process information about fundamentals, thus creating a need for financial advice in the form of investment recommendations. The behavioral constraints of conformism and robust regret aversion imply that investors disregard information contained in the price and aggregate suggested strategies by taking a weighted average, with weights summing to one. We term these implications *price information neglect* and *sum-of-weights-equals-one heuristic*. The intuition is as follows. Without the sum-of-weights-equals-one heuristic, investors would have to incorporate information contained in the price to outperform each individual suggested strategy. However, incorporating price information requires knowledge of market parameters which investors lack, and using incorrect market information leads to potentially disastrous outcomes incompatible with robust regret aversion. Thus, the investor prefers to incorporate the correct price information already captured in the strategies suggested by advisers, displaying price information neglect.

As a consequence of price information neglect and the sum-of-weights-equals-one heuristic, optimal aggregation policies disregard some of the suggested strategies and discriminate between the remaining strategies by giving higher weights to strategies suggested by high quality advisers. This occurs because the negative effect of overcounting price information outweighs the benefit of incorporating information from additional low-quality signals. The marginal benefit from increasing the weight given to a suggested strategy is decreasing and goes to zero as the weight given to that strategy goes to one. This implies that investors should consult with at least two advisers.

In the second setup with endogenous quality of advisers, investors face a cost depending on the signal precisions of their advisers. This creates a tradeoff between the accuracy of suggested strategies and the resulting cost. Our goal is to investigate how many advisers investors should consult with, how much to spend on each adviser, and how to aggregate their suggested investment strategies. We first address the latter two questions: It is optimal to spend an equal amount on each adviser and to give an equal weight to their suggested strategies given that the cost function mapping expenses on advisers to precision of signals is sufficiently convex. Convexity of the information acquisition cost function implies that homogenous advisers lead to a lower cost of a given overall precision. With sufficient convexity, this effect outweighs any benefits that could be gained by aggregating a heterogenous set of suggested strategies. We then provide a characterization of the optimal number of advisers investors should consult with. Interestingly, this number depends only on the cost function mapping expenses on advisers to precision of signals, but not on other model parameters. For the common choice of a quadratic information acquisition cost, it is optimal to consult with exactly two advisers, spend an equal amount on both of them, and equal weight their suggested strategies. Investors direct advisers to acquire less information when consulting with more advisers in order to manage overall costs. Although reducing information acquisition leads to a reduction in the total cost of acquiring information by advisers, the utility benefit of cost savings is offset by the loss in information available in the economy. Anticipating this, investors have an incentive to consult few advisers and instead direct each adviser to acquire more information.

An important insight from our analysis is that quality trumps quantity when it comes to financial advice for investors of bounded rationality. A potential policy implication is that regulatory bodies need to maintain and enhance standards for professional financial advisers and improve transparency so that investors can identify the best advisers.

Across both the exogenous and endogenous setup, the optimal number of advisers is small, but larger than one. It is not optimal to follow the suggestion of a single financial adviser, even in a model without conflicts of interest, where advisers act in the best interest of their client investors. This highlights the importance of fostering a competitive landscape for financial advice.

The remainder of this paper is organized as follows. We review the related literature in Section 1.1. In Section 2, we introduce the model of a rational expectations equilibrium economy populated by investors of bounded rationality and their financial advisers. Our main results are in Sections 3 and 4, which treat the cases of exogenous and endogenous quality of advisers. We conclude the paper in Section 5. Further discussions on modelling assumptions and market quality measures are delegated to Appendix A. Appendix B contains all proofs.

1.1 Literature Review

Our paper is first related to the emerging literature on behavioral rational expectations equilibria (Banerjee 2011; Banerjee et al. 2009; Eyster et al. 2019; Kyle and Wang 1997; Mondria et al. 2022). Kyle and Wang (1997) consider a financial market where traders may overestimate/underestimate their own and opponent's signal precisions and show that an overconfident trader can outperforms his rational opponent. This is because overconfidence acts like a commitment device in a standard Cournot duopoly model and consequently, overconfident traders trade more aggressively and his rational opponent trades relatively less aggressively. Banerjee et al. (2009) and Banerjee (2011) develop models that nest rational expectations and differences of opinion and investigate the relation between investor disagreement and market performance. Eyster et al. (2019) model that traders do not fully appreciate what prices convey about others' information and analyze whether this bias can generate large trading volume. Mondria et al. (2022) develop a model where investors cannot costlessly process price information in financial markets and show that this bounded rationality can predict price momentum and yield excessive return volatility and excessive trading volume. Our paper shares with the abovementioned literature the feature that investors are of bounded rationality. A key difference is, however, that investors of our model do not know the true market parameters and therefore cannot create investment strategies by themselves. The main problem for investors is then to optimally aggregate strategies suggested by their advisers.

Our paper is also related to the vast literature on information sales (Admati and Pfleiderer 1986, 1988, 1990; Allen 1990; Cespa 2008; García and Sangiorgi 2011; Naik 1997). Admati and Pfleiderer (1986) analyze an information sale model where there is a monopolistic seller and regular traders. The seller can get some information about the fundamental but cannot trade in a speculative market. In contrast, regular traders cannot obtain private information unless they purchase information from the monopolistic seller. The authors find that adding personalized noise to the seller's information is optimal for the seller to maximize his profits within a broad set of selling policies. Following the pioneering work of Admati and Pfleiderer (1986), some related problems that have been considered are: Indirect sale where the seller creates a portfolio based on his private information and then sells shares to traders (Admati and Pfleiderer 1988, 1990); reliability problem where the buyers are uncertain about whether the seller has superior knowledge (Allen 1990); continuous-time/dynamic settings (Cespa 2008; Naik 1997); non-competitive economies (Admati and Pfleiderer 1988; García and Sangiorgi 2011); among others. The information seller in the above literature plays a similar role as advisers in our model. For example, the seller does not invest in the market and only sells information or shares of a portfolio to investors. However, there are two main differences. First, instead of buying information from the seller, investors in our model lack full financial literacy, do not know the true market parameters, and cannot process information. Consequently, they cannot create investment strategies, thus creating a need for financial advice. Second, the focus of the discussed stream of the literature is on how the seller designs selling strategies in order to maximize his profits, while our focus is on how many advisers an investor should consult with and how much to spend on each adviser.

Our paper further contributes to the recent strand of theoretical (Colla and Antonio 2010; Han and Yang 2013; Manela 2014; Ozsoylev and Walden 2011; Walden 2019) and experimental research (Halim et al. 2019) on the implications of voluntary[1] information sharing on market

[1]Differently, Bushman and Indjejikian (1995), Indjejikian et al. (2014), and Goldstein et al. (2021) consider strategic settings where some investors have endogenous incentive to voluntarily leak their information to other

outcomes.[2] Ozsoylev and Walden (2011) analyze how the network connectedness of a large economy influences price volatility, trading volume, welfare, and other measures of interest. They find that the ex ante certainty equivalent of investors is either globally decreasing, or initially increasing and eventually decreasing in network connectedness. Manela (2014) analyzes the effect of the speed of information diffusion on the welfare of investors and shows that the value of information is hump-shaped in the diffusion speed. Walden (2019) considers a dynamic model for a rational expectations economy with decentralized information diffusion through a general network. He shows that more central investors make higher profits, and, consistent with the findings in Colla and Antonio (2010) and Ozsoylev and Walden (2011), that investors that are close to each other have more positively correlated trades. While these papers assume that information is exogenously given, Han and Yang (2013) and Halim et al. (2019) investigate the effect of social communication on market outcomes when information acquisition is endogenous. Han and Yang (2013) show that social communication reduces the endogenous fraction of informed investors and thereby harms market efficiency, reduces trading volume, and improves welfare. Halim et al. (2019) show that social communication provides an incentive for investors to free ride on other investors' information and consequently reduces the overall information in the market. Although our work is related to this literature, there are three main differences. The first main difference is that while investors in the above mentioned literature can process the information received from other investors, investors in our model are not fully financially literate and cannot process information themselves. The second difference is that while there are direct interactions among investors in the above mentioned literature, there is no direct interaction among investors in our model. Instead investors only interact with their advisers and then aggregate the suggested strategies. The third difference is that while the focus of the existing literature is on the impact of information sharing on market equilibrium, our focus is on the questions of how many financial advisers investors need and how much to spend on each adviser.

investors to increase their welfare by impacting prices.

[2]There also has been some empirical work on the effects of social communication on trading behavior of investors, see for instance, Hong et al. (2004), Hong et al. (2005), Heimer (2016), Pool et al. (2015), Ozsoylev et al. (2014), etc.

2 The Economy

Building on the finite-agent noisy rational expectations equilibrium economy of Hellwig (1980), we consider a model populated by two types of agents: Investors of bounded rationality and their fully rational financial advisers. The timeline of our model comprises three dates: $t = 0$, $t = 1$, and $t = 2$. At date $t = 0$, advisers construct investment strategies and communicate these to their client investors. At date $t = 1$, investors aggregate the strategies suggested by their advisers and submit demand schedules, noise traders trade, and the asset price is endogenously determined. At date $t = 2$, asset payoffs are realized and investors consume their resulting wealth. The order of events is described in Figure 1. Investors of our model are of bounded rationality, knowing only attributes of their advisers but not market parameters.

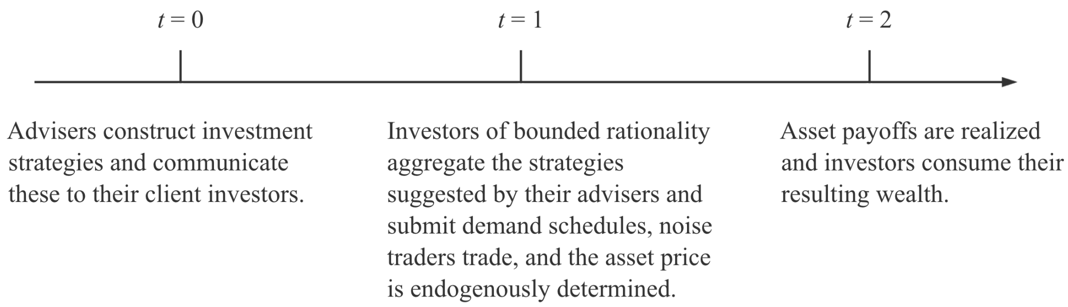

Figure 1: The timeline of the model

2.1 Assets and Investor Utility

We consider an economy where a risk-free asset and a risky asset are traded by $h \in \mathbb{N}$ investors. The risky asset has fundamental value $\theta \sim N(0, 1/\tau_\theta)$, $\tau_\theta > 0$. We suppose that preferences of the investors are represented by CARA utility functions and, without further loss of generality, that the wealth of all investors is zero. The utility investor i derives from the (stochastic) terminal wealth $W(x_i) = x_i(\theta - p)$ is thus given by

$$U\left(W(x_i)\right) = -\exp\left(-\rho x_i(\theta - p)\right),$$

where ρ is the coefficient of risk aversion[3] and p is the publicly observable price of the risky asset.

2.2 Financial Advisers

The key feature of our model is that investors do not know the true market parameters of fundamental volatility and random supply volatility, and therefore cannot create investment strategies themselves. This can be because they are not fully financially literate, or because investors are not able to search for information due to an inherent inability to do so or resource constraints. Instead of directly deciding on an investment strategy, investors consult with financial advisers and obtain their investment suggestions. Financial advisers can be traditional wealth managers, market experts, or robo-advisers.

Each investor i can consult with up to n advisers indexed by $(i,1),(i,2),\ldots,(i,n)$.[4] We assume that investors can communicate their risk preferences to their advisers[5] and that adviser (i,j) provides an investment suggestion to the investor i maximizing the expected utility of investor i.

The advisers of our model do not invest in the market themselves but only provide advice to their clients in terms of suggested investment strategies.[6] Each adviser (i,j), $i=1,\ldots,h$, $j=1,\ldots,n$, observes a private signal $y_{ij}=\theta+\epsilon_{ij}$ about the fundamental θ. The idiosyncratic

[3]All results in the paper hold for more general cases of heterogeneous risk aversion provided that the limit economy is well-defined. To simplify the setup and notation, here we only consider the case of homogeneous risk aversion.

[4]We assume that the maximal number of advisers is identical for each investor to simplify notation. Our results also hold when $n=n(i)$ differs across investors. In particular, our model allows for a subset of investors that are financially literate and construct their own investment strategies by setting $n(i)=1$ for some $i\in\mathcal{I}\subseteq\mathbb{N}$ with the interpretation that adviser and investor coincide in this case. In other words, our results also hold in a more general setting where only a fraction of investors are financially illiterate, and financially literate investors observe private signals and construct their own strategies without consulting financial advisers.

[5]Typically, advisers infer the risk preferences of their clients by asking a series of questions designed to elicit risk preferences.

[6]In practice, investors typically obtain a combination of investment strategies and information from their advisers. But the investors of our model are not fully financially literate and in particular cannot correctly interpret information. Hence, investors directly follow suggested strategies when making investment decisions.

noise $\epsilon_{ij} \sim N(0, 1/\tau_j)$ is assumed to be unbiased and independent across advisers and $\tau_j > 0$ denotes the information precision of adviser (i, j).[7] The strategy constructed by adviser (i, j) and communicated to investor i depends on both the private signal and the public price of the risky asset, i.e., $x_{ij} = x_{ij}(y_{ij}, p)$. To prevent prices from fully revealing, there is per-capita random supply u in the market satisfying $u \sim N(0, 1/\tau_u)$, $\tau_u > 0$. Alternatively, we can interpret $-u$ as the per-capita random demand of noise traders. We suppose that the random supply is independent of other random variables θ and ϵ_{ij}, $i = 1, \ldots, h, j = 1, \ldots, n$.

We assume that a single adviser does not impact prices by suggesting an investment strategy and therefore does not consider price impact. To justify this assumption, we adopt the setting of a large economy by letting $h \to \infty$ as in Hellwig (1980).

2.3 Investors of Bounded Rationality

Investors receive strategies suggested by their advisers and then aggregate these strategies to determine their personal investment strategy maximizing expected utility. A fully rational investor with complete knowledge of how advisers suggest strategies and true market parameters would first infer the signals observed by the advisers and then derive the optimal investment strategy based on all available information. Assuming linearity, the resulting rationally optimal investment strategy would be of the form

$$\sum_{j=1}^{n} a_{ij} x_{ij} - \varphi_i p \tag{1}$$

for some $a_{ij} \geq 0$ and $\varphi_i \in \mathbb{R}$ depending on the signal precision of advisers $(\tau_j)_{j=1,\ldots,n}$ as well as the market parameters τ_θ and τ_u, see Lemma 1 in the next section for details. If investors were fully rational, the setting where advisers communicate strategies is essentially equivalent

[7] Our results also hold under a more general adviser pool structure where advisers can suggest strategies to multiple investors and the adviser pools of any two investors may be different provided that there are infinitely many advisers, finitely many types of signal precisions, and that there is an upper bound on the number of advisers per investor. These assumptions assure that the limiting equilibrium is well defined and takes a linear form of the fundamental θ and the random supply u as in (6). For example, our results also apply to the setting where multiple investors share the same adviser pool. In this case, the number of advisers can be smaller than the number of investors. Furthermore, all results in the paper hold for more general cases of heterogenous signal precision across adviser pools, i.e., when the information precision of adviser (i, j) depends on both i and j.

to models of information sharing studied in the literature (Han and Yang 2013; Ozsoylev and Walden 2011).

However, investors in our model are not fully rational and they in particular do not know the values of the market parameters τ_θ and τ_u. We assume that investors in our model linearly aggregate suggested strategies and the price as in (1), but under the following bounded rationality constraints.

Definition 1. *An aggregation policy* $((a_{ij})_{j=1,\dots,n}, \varphi_i) \in \mathbb{R}^n \times \mathbb{R}$ *is called admissible under bounded rationality constraints if*

(i) Lack of financial literacy: the coefficients $(a_{ij})_{j=1,\dots,n}$ *and* φ_i *depend only on the signal precision of their advisers* $(\tau_j)_{j=1,\dots,n}$*, but not on the market parameters* τ_θ *and* τ_u*.*

(ii) Conformism: the investor will buy (or sell) the stock whenever all advisers suggest to buy (or sell) the stock.

(iii) Robust regret aversion: the aggregated strategy (1) must not be dominated by the suggested strategy of any single adviser for any $(\widetilde{\tau}_\theta, \widetilde{\tau}_u)$*. Formally, for any* $\widetilde{\tau}_\theta, \widetilde{\tau}_u > 0$*,*

$$\mathbb{E}^{\widetilde{\tau}_\theta,\widetilde{\tau}_u}\left[U\left(W\left(\sum_{j=1}^{n} a_{ij}x_{ij} - \varphi_i p\right)\right)\right] \geq \max_{1\leq j\leq n} \mathbb{E}^{\widetilde{\tau}_\theta,\widetilde{\tau}_u}\left[U\left(W\left(x_{ij}\right)\right)\right], \tag{2}$$

where $\mathbb{E}^{\widetilde{\tau}_\theta,\widetilde{\tau}_u}$ *denotes the expectation operator given market parameters* $\widetilde{\tau}_\theta, \widetilde{\tau}_u > 0$ *that do not necessarily coincide with the physical market parameters* τ_θ *and* τ_u *unknown to the investor.*

The set of aggregation policies admissible under bounded rationality constraints is denoted by $\mathcal{A}$*.*

The first constraint of bounded rationality states that investors do not know or understand the market parameters τ_θ and τ_u. We term this *lack of financial literacy*. However, we for now assume that investors know the signal precision of their advisers $(\tau_j)_{j=1,\dots,n}$. A discussion of the case where investors do not have full knowledge of the signal precision of their advisers is postponed to the appendix. The second constraint of bounded rationality states that, investors exhibit *conformism* (Cialdini and Goldstein (2004)): Investors do not short (long) the stock when all advisers recommend to take a long (short) position. The third constraint is crucial and states that the investor wants to make sure that the chosen strategy is not dominated by the strategy suggested by a single adviser. This constraint represents a form of *robust regret aversion*, robust because we require it to hold for *any* market parameters unknown to the investors.

3 Exogenous Quality of Advisers

In this section, we consider the setting where the number n of financial advisers and their quality, in terms of signal precision $(\tau_j)_{j=1,\dots,n}$, are exogenously given. We first introduce the notion of an *equilibrium with exogenous quality of advisers* for the context of our model populated by investors of bounded rationality and their financial advisers. Compared with traditional models of rational expectations equilibrium economies, the key problem to solve is how investors aggregate strategies suggested by their advisers.

Definition 2. *An equilibrium with exogenous quality of advisers is a tuple* $((x_{ij}, a^*_{ij}, \varphi^*_i)_{i=1,\dots,\infty,j=1,\dots,n}, p)$ *of strategies suggested by the advisers, aggregation policies in terms of the coefficients in (1), and the price, such that*

(i) Advisers maximize the expected utility of investors: For each i and j, x_{ij} maximizes the expected utility conditional on the private signal y_{ij} and price p, i.e.,

$$x_{ij}(y_{ij}, p) \in \arg\max_x \mathbb{E}[-\exp(-\rho x(\theta - p)) \, |y_{ij}, p].$$

(ii) Investors optimally aggregate suggested strategies under the constraints of bounded rationality: For each i, the aggregation policy $\left((a^*_{ij})_{j=1,\dots,n}, \varphi^*_i\right) \in \mathcal{A}$ is admissible under bounded rationality constraints and

$$\left((a^*_{ij})_{j=1,\dots,n}, \varphi^*_i\right) \in \underset{((a_{ij})_{j=1,\dots,n}, \varphi_i) \in \mathcal{A}}{\arg\max} \mathbb{E}^{\widetilde{\tau}^i_\theta, \widetilde{\tau}^i_u}\left[U\left(W\left(\sum_{j=1}^n a_{ij}x_{ij} - \varphi_i p\right)\right)\right]. \quad (3)$$

(iii) The market clears:

$$\lim_{h\to\infty} \frac{1}{h} \sum_{i=1}^h \left(\sum_{j=1}^n a^*_{ij} x_{ij}(y_{ij}, p) - \varphi^*_i p\right) = u.$$

Condition (i) states that advisers act in the best interest of their client investors by maximizing their ex ante expected utility given their own private signal and the price.[8] Condition (ii) describes how investors aggregate the suggested strategies communicated to them by their advisers. Condition (iii) specifies the market-clearing rule, i.e., the demand equals the supply. We will show later that an equilibrium with exogenous quality of advisers exists, is unique within the family of linear equilibria, and that the optimal aggregation policies maximizing (3) are independent of $\widetilde{\tau}_\theta^i$ and $\widetilde{\tau}_u^i$ and satisfy

$$\left((a_{ij}^*)_{j=1,\ldots,n}, \varphi_i^*\right) \in \underset{((a_{ij})_{j=1,\ldots,n},\varphi_i)\in\mathcal{A}}{\arg\max} \mathbb{E}^{\widetilde{\tau}_\theta,\widetilde{\tau}_u}\left[U\left(W\left(\sum_{j=1}^n a_{ij}x_{ij} - \varphi_i p\right)\right)\right]$$

for *any* $\widetilde{\tau}_\theta, \widetilde{\tau}_u > 0$.

We characterize equilibria with exogenous quality of advisers via the following three steps. The first step is to characterize the strategies suggested by advisers following standard arguments. The second step is to show that

$$\mathcal{A} \subseteq \left\{((a_j)_{j=1,\ldots,n}, \varphi) \,\middle|\, a_j \geq 0, \sum_{j=1}^n a_j = 1, \varphi = 0\right\}. \tag{4}$$

Finally, the third step is to show that there exist unique aggregation policies (a_{ij}^*) which are the same across investors and depend only on the signal precisions of advisers such that (3) holds with $\varphi_i^* = \varphi_i = 0$. Substituting the optimal weights into the equilibrium strategies and the equilibrium price obtained in the first step, we will get a unique equilibrium with exogenous quality of advisers.

3.1 Advisers' Suggested Strategies

As in the majority of the literature, we herein focus on *linear* equilibria, i.e., equilibria where strategies suggested by advisers are linear functions of the signal and price, and prices are linear in the signals and per-capital supply. Following the analysis in Hellwig (1980), Ozsoylev and Walden (2011), and Han and Yang (2013), we can infer the following convergence result as h increases to infinity. Due to the homogeneity of risk aversion and signal precisions of advisers

[8] One of the reasons why advisers make recommendations that are in the best interest of investors is due to the potential benefits in terms of their reputation and influence.

across investors, the optimal weights in Condition (ii) of Definition 2 will be homogenous. We will thus first assume and later verify that the optimal aggregation policies across investors are the same in equilibrium.

Suppose all investors follow the aggregation policy given by $(a_1, ..., a_n)$ and φ. Let $a = \sum_{j=1}^{n} a_j$ and let

$$\Delta = \rho^{-1} \sum_{j=1}^{n} a_j \tau_j \tag{5}$$

be the *risk adjusted average signal precision* in the economy. If $\Delta + \frac{\rho\varphi + a\tau_\theta}{a\Delta\tau_u + \rho} = 0$, there does not exist an equilibrium in the limit economy where $h \to \infty$.[9] Otherwise, the sequence of equilibrium prices of finite-agent economies converges in probability to[10]

$$p = \frac{1}{\Delta + \frac{\rho\varphi + a\tau_\theta}{a\Delta\tau_u + \rho}} (\Delta\theta - u). \tag{6}$$

In the limit of a large economy, the strategy suggested by adviser (i, j) is equal to

$$x_{ij}(y_{ij}, p) = \frac{\mathbb{E}[\theta | y_{ij}, p] - p}{\rho \operatorname{Var}[\theta | y_{ij}, p]} = \rho^{-1} \left(\tau_j y_{ij} - \left(\tau_j + \frac{\rho(\tau_\theta - \varphi\Delta\tau_u)}{a\Delta\tau_u + \rho} \right) p \right). \tag{7}$$

The first equality is the standard mean-variance portfolio strategy one obtains due to the CARA-normality setting (see, e.g., Equations (6) and (11) in Grossman (1976)). The second equality follows from (6) and the projection theorem for normal random variables.

Additional computations yield the ex ante expected utility of the suggested strategy x_{ij} under the possibly erroneous beliefs on market parameters $\widetilde{\tau}^i_\theta, \widetilde{\tau}^i_u > 0$:[11]

$$\begin{aligned} \mathbb{E}^{\widetilde{\tau}^i_\theta, \widetilde{\tau}^i_u}[U(W(x_{ij}))] &= \mathbb{E}^{\widetilde{\tau}^i_\theta, \widetilde{\tau}^i_u}[-\exp(-\rho x_{ij}(y_{ij}, p)(\theta - p))] \\ &= -\sqrt{\frac{\operatorname{Var}^{\widetilde{\tau}^i_\theta, \widetilde{\tau}^i_u}[\theta | y_{ij}, p]}{\operatorname{Var}^{\widetilde{\tau}^i_\theta, \widetilde{\tau}^i_u}(\theta - p)}} \\ &= -\left(\operatorname{Var}^{\widetilde{\tau}^i_\theta, \widetilde{\tau}^i_u}(\theta - p)(\widetilde{\tau}^i_\theta + \Delta^2 \widetilde{\tau}^i_u + \tau_j) \right)^{-\frac{1}{2}}, \end{aligned} \tag{8}$$

[9] Intuitively, when $\Delta + \frac{\rho\varphi + a\tau_\theta}{a\Delta\tau_u + \rho} = 0$, the aggregated strategy is insensitive to the price p, yielding an equilibrium price with infinite variance and price that converges to positive or negative infinity.

[10] In fact, it holds that $a = 1$ and $\varphi = 0$, so that $\Delta + \frac{\rho\varphi + a\tau_\theta}{a\Delta\tau_u + \rho} \neq 0$, under the bounded rationality constraints in Definition 1; see Proposition 1. Moreover, recall that we assumed that all random variables have mean zero for notational convenience. Hence, there is no intercept in price function p.

[11] See also the proof of Lemma 2 in Rahi and Zigrand (2018).

where $\mathrm{Var}^{\widetilde{\tau}^i_\theta,\widetilde{\tau}^i_u}$ denotes the variance operator given the market parameters $\widetilde{\tau}^i_\theta$ and $\widetilde{\tau}^i_u$. We remark that when assessing the ex-ante expected utility under individual beliefs on market parameters $\widetilde{\tau}^i_\theta$ and $\widetilde{\tau}^i_u$, the investor assumes that advisers use those same beliefs to derive their suggested strategies.

3.2 Aggregation Policies under Bounded Rationality Constraints

The following proposition shows how the constraints of bounded rationality (Definition 1) translate to formal constraints on the coefficients of aggregation policies.

PROPOSITION **1.** *The set of aggregation policies admissible under bounded rationality constraints satisfies* (4)*, i.e.,*

$$\mathcal{A} \subseteq \left\{ ((a_j)_{j=1,\ldots,n}, \varphi) \,\middle|\, a_j \geq 0, \sum_{j=1}^{n} a_j = 1, \varphi = 0 \right\}.$$

Proposition 1 shows that bounded rationality constraints have three behavioral implications on admissible aggregation policies: (i) *directional adherence* to suggested strategies, $a_{ij} \geq 0$; (ii) the *sum-of-weights-equals-one heuristic*,[12] $\sum_{j=1}^{n} a_{ij} = 1$; and (iii) *price information neglect*,[13] $\varphi_i = 0$. To better understand these implications, it is useful to consider a fully rational investor that infers signals y_{ij} from suggested strategies x_{ij}.

LEMMA **1.** *The investment strategy of a fully rational investor able to infer signals y_{ij} from suggested strategies x_{ij}, $j = 1, \ldots, n$, is given by*

$$\frac{\mathbb{E}[\theta|y_{i1}, ..., y_{in}, p] - p}{\rho\,\mathrm{Var}[\theta|y_{i1}, ..., y_{in}, p]} = \rho^{-1} \left(\sum_{j=1}^{n} \tau_j y_{ij} - \left(\frac{\rho(\tau_\theta - \varphi \Delta\tau_u)}{a\Delta\tau_u + \rho} + \sum_{j=1}^{n} \tau_j \right) p \right). \tag{9}$$

Using (7), (9), and the relation

$$\sum_{j=1}^{n} a_{ij} x_{ij} - \varphi_i p = \frac{\mathbb{E}[\theta|y_{i1}, ..., y_{in}, p] - p}{\rho\,\mathrm{Var}[\theta|y_{i1}, ..., y_{in}, p]},$$

[12]The sum-of-weights-equals-one heuristic has been widely used in the social learning literature, for example, Degroot (1974), DeMarzo et al. (2003), Golub and Jackson (2010, 2012), Jadbabaie et al. (2012).

[13]Price information neglect has been assumed in Eyster et al. (2019) to explain trading volume, where traders do not perceive the information content of others' behavior and neglect disagreements in traders' beliefs.

we can infer that the aggregation policy of a fully rational investor satisfies $a_{ij} = 1$ for all j and $\varphi_i = -(n-1)(\tau_\theta - \varphi\Delta\tau_u)/(\Delta\tau_u + \rho)$ in (1). A fully rational investor gives unit weight to each of the suggested strategies in order to optimally use the information contained in the signals. In particular, a fully rational investor increases his position by one if the suggested strategy of a single advisers x_{ij} goes up by one. In the presence of multiple advisers ($n \geq 2$), such behavior potentially overcounts the affect of the common component of the fundamental reflected in the price. To correct for this bias, the fully rational investor will adjust the price with a loading coefficient φ_i.

Note that the aggregation policy adopted by a fully rational investor depends on the true market parameters τ_θ and τ_u. This policy can thus not be adopted by the investors of bounded rationality of our model, whose aggregation policies can only depend on the signal precision of their advisers (Definition 1, Constraint (i)). Directional adherence to suggested strategies is a direct consequence of Constraint (ii) in Definition 1 stating that the investor buys (sells) the stock whenever all advisers suggest to buy (sell) the stock. Next, recall that robust regret aversion (Definition 1, Constraint (iii)) requires that the aggregated strategy must outperform the suggested strategy of any single adviser for any value of the market parameters $\widetilde{\tau}_\theta$ and $\widetilde{\tau}_u$. Together with the requirement that aggregation policies are independent of market parameters, this implies the sum-of-weights-equals-one heuristic and price information neglect. To see this, note that the market parameters enter each adviser's suggested strategy (7) through the common factor $\frac{\tau_\theta - \varphi\Delta\tau_u}{a\Delta\tau_u + \rho}$ multiplied to the price p. Without the sum-of-weight-equals-one heuristic, i.e., when $\sum_{j=1}^{n} a_{ij}$ does not equal to one, the aggregation of suggested strategies $\sum_{j=1}^{n} a_{ij}x_{ij}$ needs to be adjusted by the term $\varphi_i p$ depending on the price in order to outperform any single suggested strategy under any market conditions. The correct adjustment requires knowledge of market parameters, which investors of bounded rationality do not have. Moreover, given that the investor follows the sum-of-weights-equals-one heuristic and the suggested strategy of any single adviser puts the same weight on the price, it is optimal for investors with robust regret aversion to display price information neglect because the aggregation of suggested strategies $\sum_{j=1}^{n} a_{ij}x_{ij}$ already contains the same market parameter dependent factor applied to the price as any single suggested strategy does.

We summarize the conclusions of Proposition 1 for the context of an equilibrium as follows. Investors of bounded rationality aggregate suggested strategies under the following three behavioral patterns: directional adherence, sum-of-weights-equals-one heuristic, and price information neglect. That is, after observing the strategies suggested by his advisers $(x_{ij})_{j=1,\ldots,n}$, investor i decides on the weights $a_{ij} \geq 0$ satisfying the sum-of-weight-equals-one constraint $\sum_{j=1}^{n} a_{ij} = 1$ and then aggregates suggested strategies to

$$x_i^* := \sum_{j=1}^{n} a_{ij} x_{ij}.$$

The investors determines the weights $a_{ij} \geq 0$ in order to maximize his expected utility of the weighted strategy x_i^* under the market parameters $\widetilde{\tau}_\theta^i$ and $\widetilde{\tau}_u^i > 0$ as in (3).

We remark that some financial advisers do not only provide investment suggestions but also directly implement investment strategies on behalf of their clients. This is in particular the case for the emerging industry of robo-advisers, see, e.g., Capponi et al. (2022); D'Acunto et al. (2019); D'Acunto and Rossi (2021); Dai et al. (2021); Liang et al. (2023) for a recent literature discussing the interaction between robo-advisers and their human clients. This setting can also be covered by our model by interpreting $a_{ij}x_{ij}$ as the amount investor i transfers to adviser (i, j) which is then invested in the risky asset by the adviser on behalf of the investor.

3.3 Optimal Aggregation of Suggested Strategies

This section is concerned with the optimal aggregation of strategies suggested by advisers under the constraints of bounded rationality. We first present an important result on the ex ante expected utility of the aggregated strategy adopted by investors.

PROPOSITION **2.** *For any weight $(a_{ij})_{j=1,\ldots,n}$ with $a_{ij} \geq 0$ and $\sum_{j=1}^{n} a_{ij} = 1$, the (ex-ante) expected utility of the weighted average strategy $x_i^* = \sum_{j=1}^{n} a_{ij}x_{ij}$ is given by*

$$\begin{aligned}
\mathbb{E}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}[U(W(x_i^*))] &= -\left(\mathrm{Var}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}(\theta - p)\left(\frac{1}{\mathrm{Var}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}[\theta|p]} + \tau_i^E\right)\right)^{-\frac{1}{2}} \\
&= -\left(\mathrm{Var}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}(\theta - p)(\widetilde{\tau}_\theta^i + \Delta^2 \widetilde{\tau}_u^i + \tau_i^E)\right)^{-\frac{1}{2}} \\
&= -\left(1 + \rho\alpha_i\beta_i + \tau_i^E\gamma_i\right)^{-\frac{1}{2}},
\end{aligned} \tag{10}$$

where

$$\tau_i^E := \sum_{j=1}^{n} (2a_{ij} - a_{ij}^2)\tau_j, \tag{11}$$

and

$$\alpha_i = \frac{\Delta\rho + \rho^2/\widetilde{\tau}_u^i}{(\Delta^2\widetilde{\tau}_u^i + \Delta\rho + \widetilde{\tau}_\theta^i)^2}, \quad \beta_i = \frac{\widetilde{\tau}_\theta^i}{\Delta\widetilde{\tau}_u^i + \rho}, \quad \gamma_i = \frac{\widetilde{\tau}_\theta^i + (\Delta\widetilde{\tau}_u^i + \rho)^2/\widetilde{\tau}_u^i}{(\Delta^2\widetilde{\tau}_u^i + \Delta\rho + \widetilde{\tau}_\theta^i)^2}. \tag{12}$$

Considering (8), (10), and (11), we find that the expected utility resulting from x_i^* remains the same when replacing the signal precision τ_j of each adviser (i,j) with τ_i^E. We term τ_i^E the *equivalent signal precision*. While the strategy suggested by the adviser (i,j) enters the aggregated strategy with weight a_{ij}, the signal precision of adviser (i,j) enters into the equivalent signal precision with a weight $(2a_{ij} - a_{ij}^2)$. This term is increasing in a_{ij} since $a_{ij} \in [0,1]$ due to directional adherence and the sum-of-weights-equals-one heuristic. Moreover, the higher the signal precision of advisers, the higher the equivalent signal precision τ_i^E. Remarkably, the equivalent signal precision τ_i^E is only a function of the weights $(a_{ij})_{j=1,\dots,n}$ and the signal precisions $(\tau_j)_{j=1,\dots,n}$ of their advisers, independent of other model parameters, especially of $\widetilde{\tau}_\theta^i$ and $\widetilde{\tau}_u^i$.

Since we consider the limit of a large economy, any particular investor's decision has no impact on the price p and the risk adjusted average signal precision Δ. These quantities are endogenously determined in equilibrium. This fact together with (10) and (11) imply that the optimal aggregation of suggested strategies can be determined by maximizing τ_i^E. Specifically, investors maximize the expected utility of the weighted strategies by choosing $(a_{ij}^*)_{j=1,\dots,n}$ which solves the following maximization problem:

$$(a_{ij}^*)_{j=1,\dots,n} \in \operatorname*{arg\,max}_{a_{ij}, j=1,\dots,n} \sum_{j=1}^{n} (2a_{ij} - a_{ij}^2)\tau_j \quad \text{s.t.} \sum_{j=1}^{n} a_{ij} = 1, a_{ij} \geq 0. \tag{13}$$

Next, we show how investors optimally aggregate the suggested strategies from their advisers.

Proposition 3. *The following hold.*

(i) Suppose $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n$*, and let*

$$t = \max\left\{ j \,\middle|\, 2 \leq j \leq n, 1 + \sum_{\ell=1}^{j-1} \frac{\tau_j}{\tau_\ell} > j - 1 \right\}.$$

Then $t \geq 2$, the optimal solution to the optimization problem (13) is unique and given by

$$a_{ij}^* = \frac{a_{it}^*\tau_t + \tau_j - \tau_t}{\tau_j} = 1 - \frac{(t-1)\frac{\tau_t}{\tau_j}}{1 + \sum_{\ell=1}^{t-1}\frac{\tau_t}{\tau_\ell}}, \quad j = 1, ..., t-1;$$

$$a_{it}^* = \frac{1 + \sum_{\ell=1}^{t-1}\frac{\tau_t - \tau_\ell}{\tau_\ell}}{1 + \sum_{\ell=1}^{t-1}\frac{\tau_t}{\tau_\ell}} = 1 - \frac{t-1}{1 + \sum_{\ell=1}^{t-1}\frac{\tau_t}{\tau_\ell}};$$

$$a_{ij}^* = 0, j = t+1, ..., n.$$

The solution satisfies $a_{i1}^* \geq a_{i2}^* \geq \cdots \geq a_{it}^* > 0$, where the inequality becomes an equality if and only if the corresponding two signal precisions are identical. In particular, when $\tau_1 = \tau_2 = \cdots = \tau_n$, it holds that $a_{i1}^* = a_{i2}^* = \cdots = a_{in}^* = 1/n$.

(ii) The optimal value $\sum_{j=1}^n (2a_{ij}^*(\boldsymbol{\tau}_i) - (a_{ij}^*(\boldsymbol{\tau}))^2)\tau_j$ of (13) is increasing in τ_j for any j, where $\boldsymbol{\tau} = (\tau_1, ..., \tau_n)$. Moreover, if $a_{ij}^* > 0$, it is strictly increasing in τ_j.

The analysis of the expressions (6) and (7) together with Propositions 1 and 3 lead to the existence of a unique equilibrium.

Proposition **4.** *There exists a unique equilibrium with exogenous quality of advisers. Moreover, the optimal aggregation policies maximizing (3) are the same across investors, and are independent of $\widetilde{\tau}_\theta^i$ and $\widetilde{\tau}_u^i$.*

We close with a discussion of features of the optimal aggregation policy in equilibrium. First, for the special case of $n = 2$, we can get the explicit solution $a_{ij}^* = \tau_j/(\tau_1 + \tau_2)$, $j = 1, 2$. That is, the investor should consult both advisers no matter how large the difference in the precision between them, and the optimal weights are in proportion to advisers' signal precisions.

Second, the optimal weights given to the strategies suggested by advisers with low signal precision is zero when their signal precision is low relative to the high-precision advisers. In other words, investors cannot always benefit from more consultations.[14] Consulting with an additional adviser is only beneficial if his signal precision is in a similar range or higher than that

[14]This is in contrast to the setting where agents learn about an unknown state by observing several conditionally independent signals. In such a setting, agents always put a positive weight to each signal even with very low precision (refer to the example on page 378 in Vives (2008)). This is because any additional signal, even with a very low precision, can always improve the estimate. This is not the case in our optimal aggregation problem because there is the issue of overcounting the price.

of the advisers the investor is already consulting. In fact, for any signal precision $(\tau_j)_{j=1,\cdots,n}$, there exists a threshold $\hat{\tau}$ such that the optimal weight $a^*_{ij} > 0$ if and only if $\tau_j \geq \hat{\tau}$, and $a^*_{ij} = 0$ if and only if $\tau_j < \hat{\tau}$.

Recall from Proposition 1 that investors of bounded rationality display the sum-of-weights-equals-one heuristic and price information neglect. For such an investor, the cost of overcounting the impact of the price p may outweigh the benefit of the extra information content carried by the corresponding suggested strategy, and this is aggravated when the signal precision of the adviser is comparatively small. It is thus optimal to disregard some of the suggested strategies completely, and to discriminate between the remaining strategies by giving higher weights to strategies suggested by advisers with higher signal precision.

Third, we have $t = n$ exactly when $\tau_n > \frac{n-2}{n-1}\frac{n-1}{\sum_{\ell=1}^{n-1}\frac{1}{\tau_\ell}}$.[15] Therefore, a given investor i should consult with all of his advisers if and only if the difference between the highest precision and the other precisions is small. In particular, when $\tau_1 = \tau_2 = \cdots = \tau_{n-1}$, then $t = n$ if and only if $\tau_n > \frac{n-2}{n-1}\tau_{n-1}$. For a^*_{in} to be positive, τ_n needs to be close to τ_{n-1}, especially when n is large. This is because the marginal benefit from increasing weight given to a suggested strategy is decreasing in the weight that has already been given. If the precision of one additional signal is not too low compared with those the investor has already consulted, the benefit of the extra information conveyed by this signal outweighs the cost of overcounting the price and consequently, the additional suggested strategy with this signal will receive a positive weight. We remark that when the signal precisions are identical, i.e., $\tau_1 = \cdots = \tau_n$, the optimal weight is uniform: $a^*_{i1} = \cdots = a^*_{in} = 1/n$.

Fourth, when $\tau_2 = \tau_3 = \cdots = \tau_n$, then $t = n$ for any $\tau_1 \geq \tau_n$. That is, if there is one star adviser and all other advisers share the same lower signal precision, the investors should consult with all advisers no matter how large the difference in precision between the star adviser and all others. In this case,

$$a^*_{i1} = \frac{(n-1)(\tau_1 - \tau_n) + \tau_n}{(n-1)\tau_1 + \tau_n}, \quad a^*_{i2} = \cdots = a^*_{in} = \frac{\tau_n}{(n-1)\tau_1 + \tau_n}.$$

Letting $n \to \infty$ and keeping $\tau_1 > \tau_n$ constant, we have $a^*_{i1} \to \frac{\tau_1 - \tau_n}{\tau_1} > 0$ and $a^*_{in} \to 0$ no matter how close τ_n is to τ_1. That is, one should give a strictly positive weight to one's most trusted

[15] We remark that the threshold $\frac{n-2}{n-1}\frac{n-1}{\sum_{\ell=1}^{n-1}\frac{1}{\tau_\ell}}$ is a multiple of the harmonic mean of the $(n-1)$ signal precisions.

adviser even in the limit where one is able to consult with an arbitrary number of advisers.

Fifth, since $t \geq 2$, there are at least two positive components in the optimal weight. In other words, investors should consult at least two advisers even if the signal precision of the best advisers is much larger than that of the second best adviser. The marginal benefit from increasing the weight given to a suggested strategy converges to zero as the weight given to that strategy goes to one. Compared with the situation where all weight is given to the strategy suggested by the adviser with highest precision, an investor can thus increase his expected utility by reducing this weight by a small number and correspondingly increasing the weight given to the strategy suggested by another adviser. This finding is due to the independence of signals observed by different advisers. This explanation also applies to the cases of $n = 2$ and $\tau_2 = \tau_3 = \cdots = \tau_n$ discussed in the first and fourth part respectively.

Sixth, for the positive components of the optimal solution, the higher signal precision, the larger optimal weight. This is intuitive and reasonable.

Finally, Part (ii) of Proposition 3 shows the intuitive result that the expected utility of an investor increases in the precision of the advisers. Consider the situation where advisers are relatively homogenous such that the investor consults with all of them. Increasing the precision of a given adviser might lead to a situation where the investor now disregards some of the suggestions. The proposition shows that, in this scenario, the resulting welfare of the investor after the increase in precision of a single adviser is still superior than in the original setting where the investor consults with all advisers.

4 Endogenous Quality of Advisers

In this section, we consider the case where number and quality of advisers are determined endogenously. Advisers provide investment strategies to their clients, and in return investors reimburse their advisers with a consultation fee. Investors decide on how many advisers to consult and can direct each of their advisers to acquire more (or less) information on the asset by paying a higher (or lower) consultation fee. The investors thus face a cost depending on the signal precisions of their advisers and aim to optimally balance informativeness of the signal and resulting cost. We assume that the information acquisition cost function $c : [0, \infty) \rightarrow$

$[0,\infty)$ is the same for all advisers and strictly convex, strictly increasing, twice continuously differentiable, and satisfies the conditions $c(0)=0$ and $\lim_{\tau\to 0} c'(\tau)=0$ (Colombo et al. 2014; Han et al. 2016). Our goal is to explore how many advisers the investors should consult and how much to spend on each adviser.

4.1 Optimal Amount Spent on Advisers

We study endogenization of both information acquisition and the number of advisers. To this end, we need to first consider the case where only information acquisition is endogenous and the number of advisers is exogenously given. This is the mirror situation of the analysis in Section 3, where information acquisition was exogenous. After that we study the case where both information acquisition and the number of advisers are endogenous.

For a given exogenous number of advisers $r\in\mathbb{N}$, we denote by $\mathcal{N}_i$ the set of r advisers who provide suggestions to investor i.[16] Investors choose the signal precision for each of their r advisers to maximize their expected utility. Recall from Proposition 1 that the bounded rationality constraints imply the sum-of-weights-equals-one heuristic and price information neglect. Thus, according to the first expression in (10), (11) and taking into account the information acquisition cost, the expected utility of investor i choosing signal precisions τ_j, $j\in\mathcal{N}_i$, is given by

$$-\left(\mathrm{Var}^{\widetilde{\tau}^i_\theta,\widetilde{\tau}^i_u}(\theta-p)\exp\left(-2\rho\sum_{j\in\mathcal{N}_i}c(\tau_j)\right)\left(\frac{1}{\mathrm{Var}^{\widetilde{\tau}^i_\theta,\widetilde{\tau}^i_u}[\theta|p]}+\sum_{j\in\mathcal{N}_i}(2a_{ij}-a_{ij}^2)\tau_j\right)\right)^{-\frac{1}{2}}. \tag{14}$$

It follows from the law of total variance that $\mathrm{Var}^{\widetilde{\tau}^i_\theta,\widetilde{\tau}^i_u}(\theta-p)>\mathrm{Var}^{\widetilde{\tau}^i_\theta,\widetilde{\tau}^i_u}[\theta|p]$, which implies that the expected utility in (14) when setting $\tau_j=0$ is strictly greater than -1, the utility received when not investing. Thus, the expected utility investor i receives by optimally choosing signal precision $(\tau_j)_{j=1,\dots,r}$ is higher than when not investing in the economy under any beliefs on market parameters $\widetilde{\tau}^i_\theta$, $\widetilde{\tau}^i_u$.

Since this is a large economy where any individual decision of an investor has no impact on the price p and the risk adjusted average signal precision Δ_i, optimizing (14) can be simplified

[16] Recall that we assume that all investors in the economy have the same risk aversion coefficient. Thus, we only consider the case where all investors consult with the same number of advisers.

to

$$\max_{\tau_j, j\in\mathcal{N}_i} \left(\exp\left(-2\rho \sum_{j\in\mathcal{N}_i} c(\tau_j)\right) \left(\frac{1}{\mathrm{Var}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}[\theta|p]} + \sum_{j\in\mathcal{N}_i} (2a_{ij}^* - (a_{ij}^*)^2)\tau_j \right) \right), \tag{15}$$

where $(a_{ij}^*)_{j\in\mathcal{N}_i}$ are functions of $(\tau_j)_{j\in\mathcal{N}_i}$ given in Proposition 3.[17] The following proposition shows that the investor spends an equal amount on each adviser if the cost function is sufficiently convex.

PROPOSITION **5.** *Assume that either $c(\tau) = \kappa\tau^2$ with $\kappa > 0$, or that $c''(\cdot)$ is increasing and $\inf_{\tau>0} \frac{c''(\tau)\tau}{c'(\tau)} \geq r-1$. Then the optimal solution $(\tau_j^*)_{j\in\mathcal{N}_i}$ to the optimization problem (15) satisfies that $\tau_1^* = \cdots = \tau_r^*$ for any $\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i > 0$.*

The assumption of a quadratic cost function is common in the literature, see for example, Gao and Liang (2013), He et al. (2021) or Goldstein and Yang (2017). Condition $\inf_{\tau>0} \frac{c''(\tau)\tau}{c'(\tau)} \geq r-1$ requires that the cost function is sufficiently convex. For example, the condition is satisfied when the cost function takes the form of $\kappa\tau^\nu$ with $\nu \geq r$.[18] We will show in Proposition 6 that the condition will automatically hold when the cost function $c(\tau)$ takes the form of $\kappa\tau^\nu$ and the number of advisers is determined in equilibrium. Intuitively, consulting with a more homogenous group of advisers decreases information acquisition cost as well as the expected utility of the aggregation of suggested strategies.[19] If the information acquisition cost is sufficiently convex, the first effect dominates the second. Consequently, a more equal allocation of signal precision for advisers will increase investors' welfare. Remarkably, the optimality of homogeneous adviser precision does not depend on $\widetilde{\tau}_\theta^i$ and $\widetilde{\tau}_u^i$.

[17]Without loss of generality, we assume that the optimal aggregation weights associated with the optimal solution to (15) are positive, since investor's expected utility will be higher when removing one adviser with a low-precision signal and zero aggregation weight from his adviser pool.

[18]Without this convexity condition, Proposition 5 may not hold. For instance, when the cost function is linear and an equilibrium exists, we can show that any optimal solution $(\tau_j^*)_{j=1,\ldots,r}$ to the optimization problem (15) must be a corner solution, i.e., it is optimal to spend everything on a single adviser and then disregard the suggestions of all others.

[19]This can be better understood with one example. Consider the question of how one investor to allocate the total signal precision, say 2τ, to his two advisers. For simplicity, suppose the investor either allocates the total sum 2τ to one adviser, or equally allocates the total sum to his two advisers. From (11), we can see that the first setting will yield an equivalent signal precision 2τ, while the later one will yield an equivalent signal precision $3\tau/2$.

4.2 Optimal Number of Advisers

In view of Proposition 5, we will from now on consider only equilibria in which investors spend equal amounts on all advisers, i.e., $\tau_{j_1} = \tau_{j_2}$ for any $j_1, j_2 \in \mathcal{N}_i$. When all advisers (i, j), $j \in \mathcal{N}_i$, acquire a signal with precision τ, the total cost to the investor i is $rc(\tau)$. Moreover, in this case, it is optimal for the investor to aggregate suggested strategies by taking their simple average (see Proposition 3). We thus consider the simple average of suggested strategies in the remaining part of this section without further loss of generality.

Definition 3. *An equilibrium with endogenous quality of advisers is a tuple* $\left((r_i^*, \tau_i^*)_{i=1,\ldots,\infty}, (x_{ij})_{i=1,\ldots,\infty, j=1,\ldots r_i^*}, p\right)$ *of the number of advisers, signal precisions, strategies suggested by the advisers, and the price, such that*

(i) Advisers maximize the expected utility of investors: For each i *and* j, x_{ij} *maximizes the expected utility conditional on the private signal* $y_{ij}(\tau_i^*)$ *with precision* τ_i^* *and price* p, *i.e.,*

$$x_{ij}(y_{ij}(\tau_i^*), p) \in \arg\max_x \mathbb{E}[-\exp(-\rho x(\theta - p)) \,|y_{ij}(\tau_i^*), p].$$

(ii) Investors optimally choose the number and quality of advisers: For each $i = 1, \ldots, \infty$ *and given* $r_i^* \in \mathbb{N}$, τ_i^* *is the optimal precision for the advisers of investor* i, *i.e.,*

$$\tau_i^* \in \arg\max_{\tau_i > 0} \mathbb{E}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}\left[U\left(W\left(\frac{1}{r_i^*}\sum_{j=1}^{r_i^*} x_{ij}(y_{ij}(\tau_i), p)\right) - r_i^* c(\tau_i)\right)\right],$$

and r_i^* *is the optimal number of advisers for investor* i, *i.e., for any* $r \in \mathbb{N}$,

$$\sup_{\tau_i > 0} \mathbb{E}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}\left[U\left(W\left(\frac{1}{r}\sum_{j=1}^{r} x_{ij}(y_{ij}(\tau_i), p)\right) - rc(\tau_i)\right)\right]$$
$$\leq \mathbb{E}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}\left[U\left(W\left(\frac{1}{r_i^*}\sum_{j=1}^{r_i^*} x_{ij}(y_{ij}(\tau_i^*), p)\right) - r_i^* c(\tau_i^*)\right)\right].$$

(iii) The market clears, i.e.,

$$\lim_{h \to \infty} \frac{1}{h}\sum_{i=1}^{h}\left(\frac{1}{r_i^*}\sum_{j=1}^{r_i^*} x_{ij}(y_{ij}(\tau_i^*), p)\right) = u.$$

The following proposition identifies the optimal number of advisers within the framework of fully endogenous equilibria. Denote

$$A_+ = \sup_{\tau>0} \frac{\tau c'(\tau)}{c(\tau)}, \quad A_- = \inf_{\tau>0} \frac{\tau c'(\tau)}{c(\tau)}$$

and let $\lfloor b \rfloor$ and $\lceil b \rceil$ denote the maximum integer not greater than b and the minimum integer not smaller than b, respectively. We assume that $A_+ < \infty$.

PROPOSITION **6.** *For any $\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i > 0$, the optimal number of advisers in an equilibrium with endogenous quality of advisers satisfies*

$$\left\lfloor \frac{A_- + 1}{2} \right\rfloor \leq r_i^* \leq \left\lceil \frac{A_+ + 1}{2} \right\rceil, \ i = 1, \dots, \infty.$$

In particular,

- *if $A_+ = A_-$ is an odd number, then $r_i^* = (A_+ + 1)/2$, $i = 1, \dots, \infty$;*
- *if $A_+ = A_-$ is an even number, then $r_i^* \in \{\frac{A_+}{2}, \frac{A_+}{2} + 1\}$, $i = 1, \dots, \infty$.*

Furthermore, if the cost function is of the form $c(\tau) = \kappa\tau^2$ with $\kappa > 0$,[20] *then the optimal number of advisers in an equilibrium with endogenous quality of advisers is unique and equal to $r_i^* = 2$, $i = 1, \dots, \infty$.*

The first part of Proposition 6 gives a lower and upper bound on the number of advisers in an endogenous equilibrium. These depend exclusively on the structure of the information acquisition cost function. When the cost function $c(\tau)$ takes the form of $\kappa\tau^\nu$ (ν being a positive integer), it holds that $A_+ = A_- = \nu$. If ν is an odd number, then $r_i^* = (\nu + 1)/2$, while if ν is an even number, then $r_i^* = \nu/2$ or $r_i^* = \nu/2 + 1$. Recall that for a cost function of the form $c(\tau) = \kappa\tau^\nu$, we showed in Proposition 5 that the components of the optimal solution must be identical if $\nu \geq r$, which automatically holds, as indicated by Proposition 6.

Proposition 6 shows that investors typically consult with few advisers when the number and quality of advisers are endogenized. The intuition behind this is that investors generally

[20] In our case, when the cost function takes a more general form of $c(\tau) = \kappa_\ell\tau^\ell + \kappa_{\ell-1}\tau^{\ell-1} + \cdots + \kappa_1\tau$, $\ell \geq 2$, we have $r \leq \lceil \frac{\ell+1}{2} \rceil$, i.e., both endogenous information acquisition and number of advisers will lead to small consultation size not greater than $\lceil \frac{\ell+1}{2} \rceil$.

direct advisers to acquire less information when consulting with more advisers. Although reducing information acquisition leads to a reduction in the total cost of acquiring information by advisers, the utility benefit of cost savings is offset by the loss in information available in the economy. Anticipating this, investors have an incentive to consult few advisers where investors and instead direct each adviser to acquire more information.

It is remarkable that the optimal number of advisers for each investor is *independent* of the individual beliefs about market parameters $(\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i)$. However, the optimal signal precision and the resulting equilibrium price do depend on the parameters $(\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i)_i$. To show existence of equilibria with endogenous quality of advisers, we assume that the sample distribution of $\{(\widetilde{\tau}_\theta^1, \widetilde{\tau}_u^1), \ldots, (\widetilde{\tau}_\theta^h, \widetilde{\tau}_u^h)\}$ converges as $h \to \infty$.

PROPOSITION **7.** *There exists an equilibrium with endogenous quality of advisers. The equilibrium is unique if the cost function is of the form $c(\tau) = \kappa\tau^2$ with $\kappa > 0$.*

5 Conclusions

We consider a classical rational expectations equilibrium economy populated by two types of agents: Investors and their financial advisers. Investors lack full financial literacy and do not understand market parameters, thus creating a need for financial advice. Advisers provide strategy recommendations based on their private signals, which investors then aggregate under two bounded rationality constraints: conformism and robust regret aversion.

Our model can simultaneously explain the need for financial advise and why investors consult only with a small number of advisers despite the wide array of available sources. The main mechanism is as follows. The behavioral constraints of conformism and robust regret aversion imply that investors exhibit price information neglect and follow the sum-of-weights-equals-one heuristic: They disregard information contained in the price and aggregate suggested strategies by taking a weighted average, with weights summing to one. Intuitively, incorporating information contained in the price would require knowledge of market parameters, but because of robust regret aversion, the investor prefers incorporating the correct price dependence already captured in the strategies suggested by advisers. Consequently, investors of bounded rationality optimally disregard some of the suggested strategies and assigning higher weights to strategies suggested by their most trusted advisers. This selective aggregation arises because the drawback of overcounting price information outweighs the marginal benefits of including additional signals from advisers with low signal precision.

Our analysis yields two insights with potential policy implications. First, quality of financial advice is more critical than the quantity of advisers consulted. This finding stresses the need to improve professional standards, emphasizing the importance of quality and transparency in the regulating financial advisers. Second, while the optimal number of advisers is typically small, it is larger than one, highlighting the importance of fostering a competitive advisory market. Regulatory frameworks should protect individual investors from dependence on a single financial adviser and encourage a diversified and competitive market for financial advice.

Appendix

A Discussions

This appendix contains further discussions on the assumptions in the main body of the paper, and the impacts of consultation provided by advisers on market performance.

A.1 Knowledge of signal precisions of advisers

The discussion in the main body of this paper is based on the assumption that investors know the signal precisions of their advisers. In this appendix, we discuss how to optimally aggregate suggested strategies when not knowing the precision of individual advisers under a robust approach. We adapt Definition 1 of admissible aggregation policies to the setting where investors do not know the signal precisions of their advisers as follows.

Definition 4. *When not knowing the signal precision of advisers, an aggregation policy* $((a_{ij})_{j=1,\dots,n}, \varphi_i) \in \mathbb{R}^n \times \mathbb{R}$ *is called admissible under bounded rationality constraints if*

(i) Lack of financial literacy: the coefficients $(a_{ij})_{j=1,\dots,n}$ *and* φ_i *are constants independent of the signal precision of their advisers* $(\tau_j)_{j=1,\dots,n}$ *and market parameters* τ_θ *and* τ_u*.*

(ii) Conformism: the investor will buy (or sell) the stock whenever all the advisers suggest buying (or selling) the stock.

(iii) Robust regret aversion: the aggregated strategy (1) must not be dominated by the strategy suggested by any single adviser for any $(\widetilde{\tau}_\theta, \widetilde{\tau}_u)$. Formally, for any $\widetilde{\tau}_\theta, \widetilde{\tau}_u > 0$,

$$\inf_{(\tau_j)_{j=1,\ldots,n},\frac{1}{n}\sum_{j=1}^{n}\tau_j=\bar{\tau}_n} \mathbb{E}^{\widetilde{\tau}_\theta^i,\widetilde{\tau}_u^i}\left[U\left(W\left(\sum_{j=1}^{n} a_{ij}x_{ij}(y_{ij}(\tau_j),p) - \varphi_i p\right)\right)\right] \geq \max_{1\leq j\leq n} \inf_{(\tau_s)_{s=1,\ldots,n},\frac{1}{n}\sum_{s=1}^{n}\tau_s=\bar{\tau}_n} \mathbb{E}^{\widetilde{\tau}_\theta^i,\widetilde{\tau}_u^i}\left[U\left(W\left(x_{ij}(y_{ij}(\tau_j),p)\right)\right)\right]. \tag{16}$$

The set of aggregation policies admissible under bounded rationality constraints is again denoted by $\mathcal{A}$.

There are two differences compared with Definition 1: First, aggregation policies can no longer depend on the signal precision of advisers. Second, because investors do not know the signal precision of their advisers, they cannot compute the expected utility obtained by an aggregation policy even under individual beliefs on market parameters. They adopt a robust approach to evaluate performance under the worst-case consultation of signal precisions of advisers given that their average equals a certain value, $\bar{\tau}_n$. This approach is also used to adapt the definition of an equilibrium to the setting where investors do not know the signal precision of advisers.

Definition 5. *When not knowing the signal precision of advisers, an equilibrium with exogenous quality of advisers is a tuple $((x_{ij}, a_{ij}^*, \varphi_i^*)_{i=1,\ldots,\infty,j=1,\ldots,n}, p)$ of strategies suggested by the advisers, aggregation policies in terms of the coefficients in (1), and the price, such that*

(i) Advisers maximize the expected utility of investors: For each i and j, x_{ij} maximizes the expected utility conditional on the private signal y_{ij} and price p, i.e.,

$$x_{ij}(y_{ij}, p) \in \arg\max_x \mathbb{E}[-\exp\left(-\rho x(\theta - p)\right)|y_{ij}, p].$$

(ii) Investors optimally aggregate suggested strategies under the constraints of bounded rationality: For each i, the aggregation policy $\left((a_{ij}^*)_{j=1,\ldots,n}, \varphi_i^*\right) \in \mathcal{A}$ is admissible under

bounded rationality constraints and for any $((a_{ij})_{j=1,\dots,n}, \varphi_i) \in \mathcal{A}$,

$$\begin{aligned} &\inf_{(\tau_j)_{j=1,\dots,n}, \frac{1}{n}\sum_{j=1}^n \tau_j = \bar{\tau}_n} \mathbb{E}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}\left[U\left(W\left(\sum_{j=1}^n a_{ij}^* x_{ij}(y_{ij}(\tau_j), p) - \varphi_i^* p\right)\right)\right] \\ \geq &\inf_{(\tau_j)_{j=1,\dots,n}, \frac{1}{n}\sum_{j=1}^n \tau_j = \bar{\tau}_n} \mathbb{E}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}\left[U\left(W\left(\sum_{j=1}^n a_{ij} x_{ij}(y_{ij}(\tau_j), p) - \varphi_i p\right)\right)\right]. \end{aligned} \tag{17}$$

(iii) The market clears:

$$\lim_{h\to\infty} \frac{1}{h}\sum_{i=1}^h \left(\sum_{j=1}^n a_{ij}^* x_{ij}(y_{ij}, p) - \varphi_i^* p\right) = u.$$

We still have that admissible strategies under bounded rationality constraints satisfy directional adherence, the sum-of-weights-equals-one heuristic, and price information neglect.

PROPOSITION **8.** *When not knowing the signal precision of advisers, the set of aggregation policies admissible under bounded rationality constraints satisfies*

$$\mathcal{A} \subseteq \left\{((a_j)_{j=1,\dots,n}, \varphi) \,\middle|\, a_j \geq 0, \sum_{j=1}^n a_j = 1, \varphi = 0\right\}.$$

The optimal aggregation problem of an individual investor i under uncertainty about the quality of advisers is thus

$$\begin{aligned} \sup_{a_{ij}, j=1,\dots,n} \inf_{\tau_j, j=1,\dots,n} & \mathbb{E}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}\left[U\left(W\left(\sum_{j=1}^n a_{ij} x_{ij}(y_{ij}(\tau_j), p)\right)\right)\right], \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} = 1, a_{ij} \geq 0, \\ & \frac{1}{n}\sum_{j=1}^n \tau_j = \bar{\tau}_n. \end{aligned} \tag{18}$$

The following result shows that it is optimal to adopt the simple average of suggested strategies when not knowing the signal precision of the advisers.

PROPOSITION **9.** *The optimal solution to optimization problem* (18) *is unique and given by* $a_{ij}^* = 1/n$, $j = 1, \dots, n$.

We remark that the result of Proposition 9 still holds when replacing the constraint on the average signal precision in (18) by

$$\frac{1}{n}\sum_{j=1}^{n}\frac{1}{\tau_j} = K$$

for some $K > 0$. Similarly to Proposition 4, we can show that there exists a unique equilibrium with exogenous quality of advisers for the setting where advisers do not know the signal precision of their advisers. Moreover, optimal aggregation policies for (17) are identical across investors and independent of individual beliefs on market parameters $\widetilde{\tau}_\theta^i$ and $\widetilde{\tau}_u^i$.

Investors aggregate suggested strategies by giving an equal weight of $1/n$ to each of the n suggested strategies because the constraint $\frac{1}{n}\sum_{j=1}^{n}\tau_j = \bar{\tau}_n$ is symmetric in advisers' signal precision. We next discuss a more general setting where investors differentiate between advisers. Formally, we consider the constraint

$$\sum_{j=1}^{n} w_{ij}\tau_j = K,\ \ w_{ij} > 0,\ \ \sum_{j=1}^{n} w_{ij} = 1.$$

A larger $w_{ij} > 0$ reflects greater relative confidence of investor i in the suggestions of adviser (i, j).

Following a similar procedure as in the symmetric setting, we can define admissible aggregation policies under bounded rationality constraints and equilibria with exogenous quality of advisers. We can also show that bounded rationality constraints imply $\sum_{j=1}^{n} a_{ij} = 1$ and $\varphi_i = 0$, and then get a generalized version of (18):

$$\begin{aligned} \sup_{a_{ij}, j=1,\ldots,n}\ \inf_{\tau_j, j=1,\ldots,n}\ & \mathbb{E}^{\widetilde{\tau}_\theta^i,\widetilde{\tau}_u^i}\left[U\left(W\left(\sum_{j=1}^{n} a_{ij}x_{ij}(y_{ij}(\tau_j), p)\right)\right)\right] \\ \text{s.t.}\ & \sum_{j=1}^{n} a_{ij} = 1, a_{ij} \geq 0, \\ & \sum_{j=1}^{n} w_{ij}\tau_j = K, \end{aligned} \tag{19}$$

where $w_{ij} > 0$, $j = 1, \ldots, n$, with $\sum_{j=1}^{n} w_{ij} = 1$. Without loss of generality, we assume that $w_{i1} \geq w_{i2} \geq \cdots \geq w_{in}$.

PROPOSITION **10.** *The optimal solution to optimization problem* (19) *is unique and given by the following equalities*

$$\sum_{j=1}^{n}\left(1-\sqrt{1+\frac{w_{ij}}{w_{i1}}(a_{i1}^2-2a_{i1})}\right)=1,$$

$$a_{ij}=1-\sqrt{1+\frac{w_{ij}}{w_{i1}}(a_{i1}^2-2a_{i1})},\ j=2,\dots,n.$$

From Proposition 10, we can see that the optimal aggregation policies maximizing (19) are the same across investors, and are independent of $\widetilde{\tau}_\theta^i$ and $\widetilde{\tau}_u^i$. Under the above non-symmetric setting, we can also show that there exists a unique equilibrium with exogenous quality of advisers.

A.2 Impact of Consultation on Market Quality

In this appendix, we investigate how consultation with financial advisers affects the following market quality measures in an equilibrium with exogenous quality of advisers: *Price informativeness* is measured by $1/\operatorname{Var}(\theta|p)=\Delta^2\tau_u$ (Goldstein and Yang 2017; Han and Yang 2013; Ozsoylev and Walden 2011) and refers to the degree with which market prices reflect information on fundamentals. *Market liquidity* is measured by $\frac{1}{\partial p/\partial(-u)}=\Delta+\frac{\tau_\theta}{\Delta\tau_u+\rho}$ (see equation (6)). High market liquidity implies that a shock in supply or noise trading is absorbed without moving the price much and thus the market is deeper and more liquid (Goldstein and Yang 2017; Han and Yang 2013). *Return volatility* is measured by $\sqrt{\operatorname{Var}(\theta-p)}$. Here $\operatorname{Var}(\theta-p)=\left(\frac{\beta^2}{\tau_\theta}+\frac{1}{\tau_u}\right)/(\Delta+\beta)^2$, where $\beta=\frac{\tau_\theta}{\Delta\tau_u+\rho}$. These expressions can be obtained from equation (6) with $a=1$ and $\varphi=0$.

We compare the market quality measures implied by our model with the benchmark economy of Hellwig (1980). The benchmark economy is identical to ours except that advisers directly invest themselves based on the signals they observe. This allows for a fair comparison between the two economies because the precision of observed signals is identical. We have the following proposition in the case of exogenous quality of advisers.

PROPOSITION **11.** *Consider an equilibrium with exogenous quality of advisers. Compared with the benchmark economy, price informativeness is (weakly) higher and return volatility is (weakly) lower. Furthermore, market liquidity is (weakly) higher in informationally efficient* *markets, and when all advisers have relatively homogeneous signal precision, investor's expected utility is strictly higher.*[21]

Under optimal aggregation of suggested strategies signals observed with high precision receive a higher weight than in the benchmark economy, thereby incorporating more information into prices and consequently improving price informativeness and reducing return volatility. Higher price informativeness implies that prices are more indicative of the fundamental value and that uncertainty about the final payoff is consequently lower. Consequently, investors' strategies become more sensitive to the price. This implies that investors are more willing to provide liquidity, leading to a higher market liquidity. Moreover, the optimal aggregation efficiently reduces noise contained in suggested strategies, so that the resulting expected utility for a risk averse investor is higher.

B Proofs

The following lemma is used to compute the expected utility of a quadratic function (see the result on page 382 in Vives (2008) or Lemma A.1 in the Appendix in Marín and Rahi (1999)).

LEMMA **2.** *Suppose that* z *is an* n*-dimensional normal random vector with mean* 0 *and positive definite variance-covariance matrix* Σ*, and* B *is a symmetric* $n \times n$ *matrix. If the matrix* $(\Sigma^{-1} + 2B)$ *is positive definite, then* $\mathbb{E}[\exp(-z'Bz)] = (\det(I_n + 2\Sigma B))^{-\frac{1}{2}}$*, where* I_n *denotes the identity matrix in* $\mathbb{R}^n$ *and* $\det(\cdot)$ *is the determinant operator.*

Proof of Lemma 1

Follow directly from (6) and the projection theorem for normal variables. □

[21] When advisers have different signal precisions, price informativeness and market liquidity are strictly increasing and return volatility is strictly decreasing. Moreover, when considering a partial equilibrium framework (that is, when the equilibrium price is exogensously given), consultation can always improve investor's expected utility regardless of the difference between advisers' signal precisions.

Proof of Proposition 1

We first calculate the expected utility $\mathbb{E}^{\widetilde{\tau}^i_\theta,\widetilde{\tau}^i_u}\left[U\left(W\left(\sum_{j=1}^n a_{ij}x_{ij}-\varphi_i p\right)\right)\right]$. In this proof, we omit the superscripts *tilde* and i from $\widetilde{\tau}^i_\theta$ and $\widetilde{\tau}^i_u$ to simplify the notation. The strategy $\sum_{j=1}^n a_{ij}x_{ij}-\varphi_i p$ can be expressed as

$$x_i^* = \sum_{j=1}^n a_{ij}x_{ij}-\varphi_i p = \rho^{-1}\left(\bar{\tau}_i\theta+\xi_i-\left(\bar{\tau}_i+\rho\varphi_i+\frac{a_i(\tau_\theta-\varphi\Delta\tau_u)}{1+\rho^{-1}a\Delta\tau_u}\right)p\right),$$

where $\bar{\tau}_i=\sum_{j=1}^n a_{ij}\tau_j$, $\xi_i=\sum_{j=1}^n a_{ij}\tau_j\epsilon_{ij}$, $\Delta=\rho^{-1}\sum_{j=1}^n a_j\tau_j$, $a_i=\sum_{j=1}^n a_{ij}$, and $a=\sum_{j=1}^n a_j$.

We intend to apply Lemma 2 for $z=(\theta-p,\xi_i,p)'$. From the expressions $p=(\Delta\theta-u)/(\Delta+\beta)$ (see Equation (6)) and $\theta-p=(\beta\theta+u)/(\Delta+\beta)$ with $\beta:=\frac{\rho\varphi+a\tau_\theta}{a\Delta\tau_u+\rho}$, we can see that the variance-covariance matrix Σ and B are given by

$$\Sigma=\begin{pmatrix}\gamma & 0 & -\alpha\\ 0 & \mathrm{Var}(\xi_i) & 0\\ -\alpha & 0 & \frac{\Delta^2/\tau_\theta+1/\tau_u}{(\Delta+\beta)^2}\end{pmatrix},\quad B=\begin{pmatrix}\bar{\tau}_i & \frac{1}{2} & -\frac{\rho\bar{\beta}_i}{2}\\ \frac{1}{2} & 0 & 0\\ -\frac{\rho\bar{\beta}_i}{2} & 0 & 0\end{pmatrix},$$

where $\bar{\beta}_i=a_i\hat{\beta}+\varphi_i$, $\hat{\beta}=\frac{\tau_\theta-\varphi\Delta\tau_u}{a\Delta\tau_u+\rho}$,

$$\gamma=\frac{\beta^2/\tau_\theta+1/\tau_u}{(\Delta+\beta)^2}=\frac{(\rho\varphi+a\tau_\theta)^2/\tau_\theta+(a\Delta\tau_u+\rho)^2/\tau_u}{(a\Delta^2\tau_u+\Delta\rho+\rho\varphi+a\tau_\theta)^2},$$
$$\alpha=\frac{-\Delta\beta/\tau_\theta+1/\tau_u}{(\Delta+\beta)^2}=\frac{\rho(a\Delta\tau_u+\rho)(1/\tau_u-\Delta\varphi/\tau_\theta)}{(a\Delta^2\tau_u+\Delta\rho+\rho\varphi+a\tau_\theta)^2}.$$

In order to apply Lemma 2, we need that $\Sigma_i^{-1}+2B_i$ is positive definite. We first show the following claim. Suppose $z'Bz=\hat{z}'\hat{B}\hat{z}$, for some symmetric matrix $\hat{B}$, where z, $\hat{z}$ are two normal random vectors. Let Γ be invertible such that $\hat{z}=\Gamma z$ holds and let Σ and $\hat{\Sigma}$ denote the respective positive definite variance-covariance matrices of z and $\hat{z}$, respectively. Clearly, we have $\hat{\Sigma}=\Gamma\Sigma\Gamma'$. We claim that $\Sigma^{-1}+2B$ is positive definite if and only if $\hat{\Sigma}^{-1}+2\hat{B}$ is positive definite. First, from $\hat{z}'\hat{B}\hat{z}=z'\Gamma'\hat{B}\Gamma z=z'Bz$, we have $B=\Gamma'\hat{B}\Gamma$. Then it follows that $\Sigma^{-1}+2B=\Gamma'\hat{\Sigma}^{-1}\Gamma+2\Gamma'\hat{B}\Gamma=\Gamma'(\hat{\Sigma}^{-1}+2\hat{B})\Gamma$, which implies the claim.

Observe that we can alternatively write $\rho\left(\sum_{j=1}^n a_{ij}x_{ij}-\varphi_i p\right)(\theta-p)$ as $\hat{z}'\hat{B}\hat{z}$ for some normal random vector $\hat{z}$ and symmetric matrix $\hat{B}$. In fact, from the expressions $p=(\Delta\theta-$

$u)/(\Delta+\beta)$ (see Equation (6)) and $\theta - p = (\beta\theta + u)/(\Delta+\beta)$, we have

$$\begin{aligned}\rho\left(\sum_{j=1}^n a_{ij}x_{ij} - \varphi_i p\right)(\theta - p) &= \left(\bar{\tau}_i(\theta-p) + \xi_i - \rho\bar{\beta}_i p\right)(\theta - p)\\ &= \frac{1}{\Delta+\beta}\left(\bar{\tau}_i\frac{\beta\theta+u}{\Delta+\beta} + \xi_i - \rho\bar{\beta}_i\frac{\Delta\theta - u}{\Delta+\beta}\right)(\beta\theta+u)\\ &= \frac{1}{\Delta+\beta}\left(\frac{\bar{\tau}_i\beta - \rho\Delta\bar{\beta}_i}{\Delta+\beta}\theta + \frac{\bar{\tau}_i+\rho\bar{\beta}_i}{\Delta+\beta}u + \xi_i\right)(\beta\theta+u),\end{aligned}$$

which can be written as $\hat{z}'\hat{B}\hat{z}$ with $\hat{z} = (\theta, u, \xi_i)$ and

$$\hat{B} = \frac{1}{\Delta+\beta}\begin{pmatrix} \frac{(\bar{\tau}_i\beta - \rho\Delta\bar{\beta}_i)\beta}{\Delta+\beta} & \frac{\bar{\tau}_i\beta + \rho\bar{\beta}_i(\beta-\Delta)/2}{\Delta+\beta} & \frac{\beta}{2} \\ \frac{\bar{\tau}_i\beta + \rho\bar{\beta}_i(\beta-\Delta)/2}{\Delta+\beta} & \frac{\bar{\tau}_i+\rho\bar{\beta}_i}{\Delta+\beta} & \frac{1}{2} \\ \frac{\beta}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Let $\hat{\Sigma}$ denote the variance-covariance matrix of random vector $\hat{z}$. Some simple calculations give

$$\hat{\Sigma}^{-1} + 2\hat{B} = \begin{pmatrix} \tau_\theta + \frac{2(\bar{\tau}_i\beta - \rho\Delta\bar{\beta}_i)\beta}{(\Delta+\beta)^2} & \frac{2\bar{\tau}_i\beta + \rho\bar{\beta}_i(\beta-\Delta)}{(\Delta+\beta)^2} & \frac{\beta}{\Delta+\beta} \\ \frac{2\bar{\tau}_i\beta + \rho\bar{\beta}_i(\beta-\Delta)}{(\Delta+\beta)^2} & \tau_u + \frac{2(\bar{\tau}_i+\rho\bar{\beta}_i)}{(\Delta+\beta)^2} & \frac{1}{\Delta+\beta} \\ \frac{\beta}{\Delta+\beta} & \frac{1}{\Delta+\beta} & \frac{1}{\mathrm{Var}(\xi_i)} \end{pmatrix}.$$

By some simple but tedious derivations, we can show that $\hat{\Sigma}^{-1} + 2\hat{B}$ is positive definite. We omit the details here.

From the expressions $p = (\Delta\theta - u)/(\Delta+\beta)$ and $\theta - p = (\beta\theta+u)/(\Delta+\beta)$ again, we see that

$$(\theta - p, \xi_i, p)' = \begin{pmatrix} \frac{\beta}{\Delta+\beta} & \frac{1}{\Delta+\beta} & 0 \\ 0 & 0 & 1 \\ \frac{\Delta}{\Delta+\beta} & -\frac{1}{\Delta+\beta} & 0 \end{pmatrix}(\theta, u, \xi_i)'$$

is an invertible transformation, by the above claim, we know that matrix $\Sigma^{-1} + 2B$ is positive definite.

Using Lemma 2 with $z = (\theta - p, \xi_i, p)'$, and the matrices Σ, B, we obtain

$$\mathbb{E}^{\tilde{\tau}^i_\theta, \tilde{\tau}^i_u}\left[-\exp\left(-\rho\left(\sum_{j=1}^n a_{ij}x_{ij} - \varphi_i p\right)(\theta - p)\right)\right] = -(\det(I_3 + 2\Sigma B))^{-\frac{1}{2}} = -(\det(I_3 + 2B\Sigma))^{-\frac{1}{2}},$$

where

$$I_3 + 2B\Sigma = \begin{pmatrix} 1 + 2(\bar{\tau}_i\gamma + \frac{\rho\alpha\bar{\beta}_i}{2}) & \mathrm{Var}(\xi_i) & 2\phi_i \\ \gamma & 1 & -\alpha \\ -\rho\bar{\beta}_i\gamma & 0 & 1 + \rho\alpha\bar{\beta}_i \end{pmatrix}$$

with $\phi_i = -\bar{\tau}_i\alpha - \frac{\rho\bar{\beta}_i}{2}\frac{\Delta^2/\tau_\theta+1/\tau_u}{(\Delta+\beta)^2}$. Expanding the determinant $\det(I_3 + 2B\Sigma)$ along the first row yields

$$
\begin{aligned}
&\det(I_3 + 2B\Sigma) \\
&= (1 + 2\bar{\tau}_i\gamma + \rho\alpha\bar{\beta}_i)(1 + \rho\alpha\bar{\beta}_i) - \mathrm{Var}(\xi_i)\gamma + 2\phi_i\rho\bar{\beta}_i\gamma \\
&= (1 + \rho\alpha\bar{\beta}_i)^2 - (\rho\bar{\beta}_i)^2\gamma\frac{\Delta^2/\tau_\theta + 1/\tau_u}{(\Delta+\beta)^2} + 2\bar{\tau}_i\gamma(1 + \rho\alpha\bar{\beta}_i) - \mathrm{Var}(\xi_i)\gamma - 2\bar{\tau}_i\alpha\rho\bar{\beta}_i\gamma \\
&= (1 + \rho\alpha(a_i\hat{\beta} + \varphi_i))^2 - \rho^2(a_i\hat{\beta} + \varphi_i)^2\gamma\frac{\Delta^2/\tau_\theta + 1/\tau_u}{(\Delta+\beta)^2} + (2\bar{\tau}_i - \mathrm{Var}(\xi_i))\gamma \\
&= \rho^2\left(\alpha^2 - \gamma\frac{\Delta^2/\tau_\theta + 1/\tau_u}{(\Delta+\beta)^2}\right)\varphi_i^2 + \left(2\rho\alpha(1 + \rho\alpha a_i\hat{\beta}) - 2\rho^2 a_i\hat{\beta}\gamma\frac{\Delta^2/\tau_\theta + 1/\tau_u}{(\Delta+\beta)^2}\right)\varphi_i \\
&\quad + (1 + \rho\alpha a_i\hat{\beta})^2 - (\rho a_i\hat{\beta})^2\gamma\frac{\Delta^2/\tau_\theta + 1/\tau_u}{(\Delta+\beta)^2} + (2\bar{\tau}_i - \mathrm{Var}(\xi_i))\gamma \\
&= -\frac{\rho^2}{\tau_\theta\tau_u(\Delta+\beta)^2}\varphi_i^2 + 2\rho\left(\alpha - \frac{\rho a_i\hat{\beta}}{\tau_\theta\tau_u(\Delta+\beta)^2}\right)\varphi_i \\
&\quad + 1 + 2\rho\alpha\hat{\beta}a_i - \frac{(\rho a_i\hat{\beta})^2}{\tau_\theta\tau_u(\Delta+\beta)^2} + (2\bar{\tau}_i - \mathrm{Var}(\xi_i))\gamma \\
&= -\frac{\rho\alpha}{\hat{\beta}}\varphi_i^2 + 2\rho\alpha(1 - a_i)\varphi_i + 1 + \rho\alpha\hat{\beta}(2a_i - a_i^2) + \sum_{j=1}^n (2a_{ij} - a_{ij}^2)\tau_j\gamma, \qquad (20)
\end{aligned}
$$

where we use the relations

$$\alpha^2 - \gamma\frac{\Delta^2/\tau_\theta + 1/\tau_u}{(\Delta+\beta)^2} = -\frac{1}{\tau_\theta\tau_u(\Delta+\beta)^2},$$

and

$$\alpha = \frac{\rho\hat{\beta}}{\tau_\theta\tau_u(\Delta+\beta)^2}.$$

Here we assume without loss of generality that $\hat{\beta} \neq 0$ since we can exclude the values of τ_θ and τ_u for which $\hat{\beta} = 0$ in the following contradiction argument. With the substitution $a_{ij} = 1$, $a_{ij_0} = 0, j_0 \neq j$, and $\varphi_i = 0$ in (20), we can get the expected utility $1 + \rho\alpha\hat{\beta} + \tau_j\gamma$ of the suggested strategy x_{ij}.

Thus, (2) reduces to

$$-\frac{\rho\alpha}{\hat{\beta}}\varphi_i^2 + 2\rho\alpha(1 - a_i)\varphi_i + 1 + \rho\alpha\hat{\beta}(2a_i - a_i^2) + \sum_{j=1}^n (2a_{ij} - a_{ij}^2)\tau_j\gamma \geq 1 + \rho\alpha\hat{\beta} + \max_{1\leq j\leq n}\tau_j\gamma, \qquad (21)$$

i.e.,

$$-\frac{\rho\alpha}{\hat{\beta}\gamma}\varphi_i^2 + \frac{2\rho\alpha}{\gamma}(1-a_i)\varphi_i - \frac{\rho\alpha\hat{\beta}}{\gamma}(1-a_i)^2 + \sum_{j=1}^{n}(2a_{ij} - a_{ij}^2)\tau_j - \max_{1\le j\le n}\tau_j \ge 0 \tag{22}$$

Recall that the investor i wants to set the values of φ_i and a_i which are only the functions of τ_j's to make (22) holds for any values of τ_u and τ_θ. Then we can first conclude that φ_i must be zero since otherwise, we can select sufficiently small τ_θ and τ_u so that α/γ is positive, bounded, far away from zero, and $\hat{\beta}$ is positive and sufficiently small, for which (22) is impossible to hold. Thus, it must hold that $\varphi_i = 0$. Moreover, we can also conclude that the weight sum a_i must be one since otherwise, when $\alpha\hat{\beta}/\gamma$ is sufficiently large, (22) is impossible to hold. In fact, we can see that when τ_θ is large and τ_u is very small,

$$\frac{\alpha\hat{\beta}}{\gamma} = \frac{\rho(\tau_\theta/\tau_u - 2\Delta\varphi + \Delta^2\tau_u\varphi^2/\tau_\theta)}{\rho^2\varphi^2/\tau_\theta + a^2\tau_\theta + 2a\rho\varphi + a^2\Delta^2\tau_u + \rho^2/\tau_u + 2a\Delta\rho}$$

is indeed sufficiently large. Finally, from Condition (ii) in Definition 1 we can show by contradiction that each a_{ij} is non-negative. The proof is completed. □

Proof of Proposition 2

The third expression in (10) follows from the term at LHS of (21) with the setting of $\varphi_i = 0$ and $a_i = 1$. Following the similar arguments in the proof of Proposition 1, we can show that the expected utility at x_{ij} is given by

$$\mathbb{E}^{\tilde{\tau}_\theta^i, \tilde{\tau}_u^i}[-\exp(-\rho x_{ij}(\theta - p))] = -\left(1 + \rho\alpha_i\beta_i + \tau_j\gamma_i\right)^{-\frac{1}{2}}.$$

Therefore, each investor's welfare by adopting the weighted average x_i^* will be the same as that by directly following the suggested strategy by adviser (i,j) if $\tau_j = \sum_{j_0=1}^{n}(2a_{ij_0} - a_{ij_0}^2)\tau_{j_0}$. The second expression in (10) then follows from the alternative expression (8) of the expected utility at x_{ij}. Moreover, the first expression in (10) follows from the second one and the projection theorem for normal random variables. The proof is completed. □

Proof of Proposition 3

We first show Part (i). To economize the notation, here we omit the subscript i in a_{ij}, and instead consider the following constrained optimization problem:

$$\max_{a_j, j=1,\ldots,n} \sum_{j=1}^{n}(2a_j - a_j^2)\tau_j \text{ s.t. } \sum_{j=1}^{n} a_j = 1, a_j \geq 0. \tag{23}$$

and show the following result.

Suppose $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n > 0$, and let $t = \max\{j | 2 \leq j \leq n, 1 + \sum_{\ell=1}^{j-1} \frac{\tau_j - \tau_\ell}{\tau_\ell} > 0\}$, $2 \leq t \leq n$. Then the unique optimal solution to the optimization problem (23) is given by

$$a_t^* = 1 - \frac{t-1}{1 + \sum_{\ell=1}^{t-1} \frac{\tau_t}{\tau_\ell}};$$

$$a_j^* = \frac{a_t^* \tau_t + \tau_j - \tau_t}{\tau_j} = 1 - \frac{(t-1)\frac{\tau_t}{\tau_j}}{1 + \sum_{\ell=1}^{t-1} \frac{\tau_t}{\tau_\ell}}, j = 1, \ldots, t-1;$$

$$a_j^* = 0, j = t+1, \ldots, n,$$

and the solution satisfies that $a_1^* \geq a_2^* \geq \cdots \geq a_t^* > 0$, where the inequality becomes equality if and only if the corresponding two signal precisions are identical.

Proof. There exists an optimal solution $(a_j^*)_{j=1,\ldots,n}$ with $0 \leq a_j^* \leq 1$ to this constrained optimization problem since the constraint set is a bounded, closed set and the objective function is continuous. Moreover, the optimal solution is unique since the objective function is strictly convex.

We now derive the necessary conditions that the optimal solution satisfies. We claim that for any i and j with $a_j^* > 0$, it must hold that $a_i^* \tau_i - a_j^* \tau_j = \tau_i - \tau_j$. Let $0 < \delta < a_j^*$ and consider the feasible solution where the i-th component is $a_i^* + \delta$, the j-th component is $a_j^* - \delta$ and the other components equal $a_\ell^*, \ell \neq i, j$. The function value with the feasible solution is given by

$$2(a_i^* + \delta)\tau_i - (a_i^* + \delta)^2 \tau_i + 2(a_j^* - \delta)\tau_j - (a_j^* - \delta)^2 \tau_j + \sum_{\ell \neq i,j} (2a_\ell^* - (a_\ell^*)^2)\tau_\ell,$$

which achieves its maximum at $\delta = 0$. Taking derivative at $\delta = 0$ leads to the claim. The claim implies that $a_i^* \tau_i \geq a_j^* \tau_j > 0$ whenever $\tau_i \geq \tau_j$ and $a_j^* > 0$, and further that if $a_j^* > 0$, then $a_i^* \geq a_j^* > 0$ for all $i \leq j$ (otherwise, if $a_i^* < a_j^*$, then $a_i^* \tau_i - a_j^* \tau_j < a_j^*(\tau_i - \tau_j) \leq \tau_i - \tau_j$, a contradiction), and $a_i^* = a_j^*$ if and only if $\tau_i = \tau_j$. That is, if the optimal weight given to one low precision is positive, then the optimal weight given to one high precision is larger.

According to the definition of t, we have $2 \le t \le n$. We claim that $a_j^* = 0$ for all $j \ge t+1$. Otherwise, let $s = \max\{j | t+1 \le j \le n, a_j^* > 0\}$, then from the relation $a_j^*\tau_j - a_s^*\tau_s = \tau_j - \tau_s$ for $j \le s$, we have $a_j^* = \frac{a_s^*\tau_s + \tau_j - \tau_s}{\tau_j}, j = 1, \ldots, s$. By the definition of s, $a_j^* = 0$ for $j \ge s+1$. Thus,

$$\sum_{\ell=1}^{n} a_\ell^* = \sum_{\ell=1}^{s} a_\ell^* = \sum_{\ell=1}^{s} \frac{a_s^*\tau_s + \tau_\ell - \tau_s}{\tau_\ell} = 1.$$

We can solve $a_s^* = \frac{1+\sum_{\ell=1}^{s-1}\frac{\tau_s - \tau_\ell}{\tau_\ell}}{1+\sum_{\ell=1}^{s-1}\frac{\tau_s}{\tau_\ell}}$, which is nonpositive from the definition of t, but positive by the definition of s, a contradiction. Thus, $a_j^* = 0$ for all $j \ge t+1$. Similar to the above arguments, we can solve

$$a_t^* = \frac{1+\sum_{\ell=1}^{t-1}\frac{\tau_t - \tau_\ell}{\tau_\ell}}{1+\sum_{\ell=1}^{t-1}\frac{\tau_t}{\tau_\ell}} = 1 - \frac{t-1}{1+\sum_{\ell=1}^{t-1}\frac{\tau_t}{\tau_\ell}},$$

which is positive by the definition of t, and

$$a_j^* = \frac{a_t^*\tau_t + \tau_j - \tau_t}{\tau_j} = 1 - \frac{(t-1)\frac{\tau_t}{\tau_j}}{1+\sum_{\ell=1}^{t-1}\frac{\tau_t}{\tau_\ell}}$$

for $j = 1, \ldots, t-1$ as given in the lemma based on the established relation $a_j^*\tau_j - a_t^*\tau_t = \tau_j - \tau_t$. Moreover, the relation $a_1^* \ge a_2^* \ge \cdots \ge a_t^*$ follows from the result that if $a_j^* > 0$, then $a_i^* \ge a_j^* > 0$ for all $i \le j$ we have shown in the first paragraph. The last part is straightforward.

Taking the notations in Part (i), we now show Part (ii). We have

$$\frac{\partial \sum_{j=1}^{n}(2a_j^* - (a_j^*)^2)\tau_j}{\partial \tau_1} = 2a_1^* - (a_1^*)^2 + (2-2a_1^*)\frac{\partial a_1^*}{\partial \tau_1}\tau_1 + \sum_{j\neq 1}(2-2a_j^*)\frac{\partial a_j^*}{\partial \tau_1}\tau_j. \tag{24}$$

By Proposition 3 Part (i), we have

$$\begin{aligned}
\frac{\partial a_t^*}{\partial \tau_1} &= -\frac{t-1}{(1+\sum_{\ell=1}^{t-1}\frac{\tau_t}{\tau_\ell})^2}\frac{\tau_t}{\tau_1^2},\\
\frac{\partial a_j^*}{\partial \tau_1} &= \frac{\partial a_t^*}{\partial \tau_1}\frac{\tau_t}{\tau_j}, j \neq 1,\\
\frac{\partial a_1^*}{\partial \tau_1} &= \frac{\partial[(a_t^*-1)\frac{\tau_t}{\tau_1}]}{\partial \tau_1} = \frac{\partial a_t^*}{\partial \tau_1}\frac{\tau_t}{\tau_1} - (a_t^*-1)\frac{\tau_t}{\tau_1^2}.
\end{aligned}$$

As a result,

$$\begin{aligned}
(2-2a_1^*)\frac{\partial a_1^*}{\partial \tau_1}\tau_1 + \sum_{j\neq 1}(2-2a_j^*)\frac{\partial a_j^*}{\partial \tau_1}\tau_j &= \sum_{j=1}^{t}(2-2a_j^*)\frac{\partial a_t^*}{\partial \tau_1}\tau_t + (2-2a_1^*)(1-a_t^*)\frac{\tau_t}{\tau_1}\\
&= -\frac{(2t-2)(t-1)}{(1+\sum_{\ell=1}^{t-1}\frac{\tau_t}{\tau_\ell})^2}\frac{\tau_t^2}{\tau_1^2} + 2\frac{(t-1)\frac{\tau_t}{\tau_1}}{1+\sum_{\ell=1}^{t-1}\frac{\tau_t}{\tau_\ell}}\frac{t-1}{1+\sum_{\ell=1}^{t-1}\frac{\tau_t}{\tau_\ell}}\frac{\tau_t}{\tau_1}
\end{aligned}$$

$$= 0. \tag{25}$$

From (24) and (25), we obtain

$$\frac{\partial \sum_{j=1}^{n}(2a_j^* - (a_j^*)^2)\tau_j}{\partial \tau_1} = 2a_1^* - (a_1^*)^2 > 0. \tag{26}$$

The sensitivity analysis with respect to other τ_j is similar and omitted. The proof is completed. □

Proof of Proposition 4

The existence and uniqueness of equilibrium follows from replacing the weights $(a_j)_{j=1,\dots,n}$ in the expressions (6), (5) and (7) with the optimal weight solution given in Proposition 3. Clearly, the optimal aggregation policy is independent of $\widetilde{\tau}_\theta^i$ and $\widetilde{\tau}_u^i$ and the same across investors. □

Proof of Proposition 5

To economize the notation, in this proof we remove the subscript i and the superscript $*$ in a_{ij}^*. The conclusion is obvious when $r = 1$. We next assume that $r \geq 2$. Suppose $\tau_1^* \geq \tau_2^* \geq \cdots \geq \tau_r^*$ is an optimal solution to the optimization problem (15) and let the corresponding optimal weights be $a_1 \geq a_2 \geq \cdots \geq a_r$. It is clear that $a_r > 0$ since otherwise, lowering the precision τ_r^* by a sufficiently small number will reduce the information acquisition cost, and then increase the expected utility, arising a contradiction.

Taking partial derivative with respect to τ_s over (15) and let $\tau_s = \tau_s^*$ leads to

$$\begin{aligned} &-2\rho c'(\tau_s^*)\left(\frac{1}{\mathrm{Var}^{\widetilde{\tau}_\theta^i,\widetilde{\tau}_u^i}[\theta|p]} + \sum_{j=1}^{r}(2a_j - a_j^2)\tau_j^*\right) \\ &\qquad + \left.\frac{\partial[\sum_{j\neq s}^{r}(2a_j - a_j^2)\tau_j^* + (2a_s - a_s^2)\tau_s]}{\partial \tau_s}\right|_{\tau_s=\tau_s^*} = 0. \end{aligned} \tag{27}$$

It follows from the relation (26) that

$$\left.\frac{\partial[\sum_{j\neq s}^{r}(2a_j - a_j^2)\tau_j^* + (2a_s - a_s^2)\tau_s]}{\partial \tau_s}\right|_{\tau_s=\tau_s^*} = 2a_s - a_s^2, \tag{28}$$

and consequently, from (27)

$$-2\rho c'(\tau_s^*)\left(\frac{1}{\mathrm{Var}^{\widetilde{\tau}_\theta^i,\widetilde{\tau}_u^i}[\theta|p]}+\sum_{j=1}^{r}(2a_j-a_j^2)\tau_j^*\right)+2a_s-a_s^2=0. \tag{29}$$

We first show the conclusion by contradiction for general cost functions. We further remove the superscript $*$ in τ_i^* for simplifying notations. Without loss of generality, we assume that $\tau_1>\tau_2$ (otherwise if $\tau_1=\tau_2$, then we consider $\{\tau_{j_1},\tau_{j_1+1}\}$ instead of $\{\tau_1,\tau_2\}$ and a similar contradiction can be obtained, where j_1 is the smallest index such that $\tau_{j_1}\neq\tau_{j_1+1}$). Let $0<\delta<\tau_1$ and let us consider the feasible solution $(\tau_1-\delta,\tau_2+\delta,\tau_3,\dots,\tau_r)$. The corresponding function value of the objective in (15) at this feasible solution is given by

$$\begin{aligned}g(\delta):=&\exp\left(-2\rho\sum_{j\geq 3}c(\tau_j)\right)\exp\left(-2\rho(c(\tau_1-\delta)+c(\tau_2+\delta))\right)\\&\times\left(\frac{1}{\mathrm{Var}^{\widetilde{\tau}_\theta^i,\widetilde{\tau}_u^i}[\theta|p]}+\sum_{j\geq 3}^{r}(2a_j-a_j^2)\tau_j+(2a_1-a_1^2)(\tau_1-\delta)+(2a_2-a_2^2)(\tau_2+\delta)\right).\end{aligned} \tag{30}$$

Note that in (30), $(a_j)_{j=1,\dots,r}$ are the optimal weights corresponding to $(\tau_1-\delta,\tau_2+\delta,\tau_3,\dots,\tau_r)$ and depend on δ. It is clear that $g(0)\geq g(\delta)$ for any small δ by the optimality of $(\tau_1,\dots,\tau_r)$.

From (30), we have

$$\begin{aligned}&\frac{\partial g(\delta)}{\partial\delta}\Big|_{\delta=0}\\&\propto 2\rho(c'(\tau_1)-c'(\tau_2))\left(\frac{1}{\mathrm{Var}^{\widetilde{\tau}_\theta^i,\widetilde{\tau}_u^i}[\theta|p]}+\sum_{j=1}^{r}(2a_j-a_j^2)\tau_j\right)\\&\qquad-\frac{\partial\sum_{j=1}^{r}(2a_j-a_j^2)\tau_j}{\partial\tau_1}+\frac{\partial\sum_{j=1}^{r}(2a_j-a_j^2)\tau_j}{\partial\tau_2}\\&\propto 2\rho(c'(\tau_1)-c'(\tau_2))\left(\frac{1}{\mathrm{Var}^{\widetilde{\tau}_\theta^i,\widetilde{\tau}_u^i}[\theta|p]}+\sum_{j=1}^{r}(2a_j-a_j^2)\tau_j\right)-(2-a_1-a_2)(a_1-a_2)\\&\geq 2\rho c''(\tau_2)(\tau_1-\tau_2)\left(\frac{1}{\mathrm{Var}^{\widetilde{\tau}_\theta^i,\widetilde{\tau}_u^i}[\theta|p]}+\sum_{j=1}^{r}(2a_j-a_j^2)\tau_j\right)-(2-a_1-a_2)(1-a_2)\frac{\tau_1-\tau_2}{\tau_1}\\&\propto 2\rho c''(\tau_2)\tau_1\left(\frac{1}{\mathrm{Var}^{\widetilde{\tau}_\theta^i,\widetilde{\tau}_u^i}[\theta|p]}+\sum_{j=1}^{r}(2a_j-a_j^2)\tau_j\right)-(2-a_1-a_2)(1-a_2)\\&=2\rho c''(\tau_2)\tau_1\frac{2a_2-a_2^2}{2\rho c'(\tau_2)}-(2-a_1-a_2)(1-a_2)\\&=a_2(2-a_2)\frac{c''(\tau_2)\tau_1}{c'(\tau_2)}-(2-a_1-a_2)(1-a_2)\end{aligned}$$

$$> \frac{(2-a_2)(1-a_1)\frac{\tau_1}{\tau_2}}{r-1}\frac{c''(\tau_2)\tau_2}{c'(\tau_2)} - (2-a_2)(1-a_2)$$
$$\propto \frac{1}{r-1}\frac{c''(\tau_2)\tau_2}{c'(\tau_2)} - 1,$$

which is non-negative under the condition $\inf_{\tau>0}\frac{c''(\tau)\tau}{c'(\tau)} \geq r-1$ assumed in the proposition, where the second $\propto$ follows from (28), the first inequality follows from the relations

$$c'(\tau_1) - c'(\tau_2) = c''(\tau)(\tau_1-\tau_2) \geq c''(\tau_2)(\tau_1-\tau_2)$$

for some $\tau_2 < \tau < \tau_1$ (using the increasingness of $c''(\cdot)$ assumed in the proposition) and

$$a_1 - a_2 = (1-a_t)\tau_t\left(\frac{1}{\tau_2}-\frac{1}{\tau_1}\right) = (1-a_2)\tau_2\left(\frac{1}{\tau_2}-\frac{1}{\tau_1}\right) = (1-a_2)\frac{\tau_1-\tau_2}{\tau_1},$$

the first equality from (29), the second inequality from the relation $a_2 \geq \cdots \geq a_t$ and then $a_2 \geq \sum_{j=2}^{t} a_j/(t-1) \geq (1-a_1)/(r-1)$, the last $\propto$ from the relation

$$(1-a_1)\frac{\tau_1}{\tau_2} = \frac{(t-1)\frac{\tau_t}{\tau_1}}{1+\sum_{\ell=1}^{t-1}\frac{\tau_t}{\tau_\ell}}\frac{\tau_1}{\tau_2} = \frac{(t-1)\frac{\tau_t}{\tau_2}}{1+\sum_{\ell=1}^{t-1}\frac{\tau_t}{\tau_\ell}} = 1-a_2.$$

Thus, $g(\delta) > g(0)$ for sufficiently small δ. This contradicts the optimality of $(\tau_j)_{j=1,\ldots,r}$. This completes the proof of the first part.

We now show the conclusion for quadratic cost functions. Note that the cost function is of the form $c(\tau) = \kappa\tau^2$. Then it follows from (29) that

$$\frac{\tau^*_{r_1}}{\tau^*_{r_2}} = \frac{a_{r_1}(2-a_{r_1})}{a_{r_2}(2-a_{r_2})}, \forall 1 \leq r_1, r_2 \leq r.$$

Moreover, from Proposition 3, we have

$$(1-a_{r_1})\tau^*_{r_1} = (1-a_{r_2})\tau^*_{r_2}, \forall 1 \leq r_1, r_2 \leq r.$$

Combining the preceding two equalities together yields

$$a_{r_1}(1-a_{r_1})(2-a_{r_1}) = a_{r_2}(1-a_{r_2})(2-a_{r_2}), \forall 1 \leq r_1, r_2 \leq r. \tag{31}$$

Define function $f(z) = z(1-z)(2-z)$, $z \in [0,1]$. With some simple calculations, we can see that $f(0) = f(1) = 0$, $f(\cdot)$ first increases and then decreases on the interval $[0,1]$. We claim that $a_{r_1} = a_{r_2}$ for any $1 \leq r_1, r_2 \leq r$. First consider the case of $r = 2$. In this case,

$a_1 = 1 - a_2$. The claim then follows from the fact that the equality $f(z) = f(1-z)$ holds only when $z = 1/2$. Now consider $r \geq 3$, and denote $\bar{a} := a_s(1-a_s)(2-a_s)$, $s = 1, \ldots, r$. Note that the equation $f(z) = \bar{a}$ has two roots on $(0,1)$, denoted as z_- and z_+, $z_- < z_+$. Consequently, for each $1 \leq s \leq r$, either $a_s = z_-$ or $a_s = z_+$. If $a_s = z_-$ for each s, or $a_s = z_+$ for each s, then the claim follows. Now suppose $a_{r_1} = z_-$ for some r_1, and $a_{r_2} = z_+$ for some $r_2 \neq r_1$. Then since $r \geq 3$, there exists r_3 with $r_3 \neq r_2, r_3 \neq r_1$ such that $a_{r_3} = z_-$ or $a_{r_3} = z_+$. Without loss of generality, suppose $a_{r_3} = z_-$. Observe that $z_- < 1/3$ due to $r \geq 3$. With some simple calculations, we have $f(1-2z_-) > f(z_-)$, which implies that $z_+ > 1 - 2z_-$, i.e., $2z_- + z_+ > 1$, contradicting the fact $\sum_{s=1}^{r} a_s = 1$. Thus, the claim follows. As a result, it follows from Proposition 3 part (i) that $\tau_1^* = \cdots = \tau_r^*$. This completes the proof. $\square$

Proof of Proposition 6

Consider any investor i and his any possible consultation set $\mathcal{N}$ with $|\mathcal{N}| = r$, and denote $D_r^i = \mathbb{E}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}\left[U\left(W\left(\frac{1}{r}\sum_{j\in\mathcal{N}} x_{ij}(y_{ij}(\tau_i^*(r)), p)\right) - rc\left(\tau_i^*(r)\right)\right)\right]$. From (14), the expected utility of investor i D_r^i is given by

$$D_r^i = -\left[\mathrm{Var}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}(\theta - p)\exp\left(-2\rho rc(\tau_i^*(r))\right)\left(\frac{1}{\mathrm{Var}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}[\theta|p]} + \left(2 - \frac{1}{r}\right)\tau_i^*(r)\right)\right]^{-\frac{1}{2}}, \quad (32)$$

where here $\Delta = \rho^{-1}\tau_i^*(r)$, $\tau_i^*(r)$ satisfies

$$2\rho c'(\tau_i^*(r))\left(\frac{1}{\mathrm{Var}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}[\theta|p]} + \left(2 - \frac{1}{r}\right)\tau_i^*(r)\right) = \frac{2}{r} - \frac{1}{r^2}. \quad (33)$$

By letting r be a fictitious, continuous variable taking values in $[1, \infty)$, we first analyze the monotonicity of D_r^i, or equivalently,

$$\hat{D}_r^i := \exp\left(-2\rho rc(\tau_i^*(r))\right)\left(\frac{1}{\mathrm{Var}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}[\theta|p]} + \left(2 - \frac{1}{r}\right)\tau_i^*(r)\right).$$

We have

$$\frac{\partial \hat{D}_r^i}{\partial r} \propto -\left(2\rho rc'(\tau_i^*(r))\frac{\partial \tau_i^*(r)}{\partial r} + 2\rho c(\tau_i^*(r))\right)\left(\frac{1}{\mathrm{Var}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}[\theta|p]} + \left(2 - \frac{1}{r}\right)\tau_i^*(r)\right)$$
$$+ \frac{\tau_i^*(r)}{r^2} + \left(2 - \frac{1}{r}\right)\frac{\partial \tau_i^*(r)}{\partial r}$$

$$
\begin{aligned}
&= \frac{\tau_i^*(r)}{r^2} - 2\rho c(\tau_i^*(r)) \left(\frac{1}{\mathrm{Var}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}[\theta|p]} + \left(2 - \frac{1}{r}\right) \tau_i^*(r) \right) \\
&= \frac{\tau_i^*(r)}{r^2} - \frac{c(\tau_i^*(r))}{c'(\tau_i^*(r))} \left(\frac{2}{r} - \frac{1}{r^2} \right) \\
&\propto \tau_i^*(r) - \frac{c(\tau_i^*(r))}{c'(\tau_i^*(r))} (2r - 1), \qquad (34)
\end{aligned}
$$

where the two equalities follow from (33). Thus, the lower and upper bound in the proposition follow from (34).

We now consider the case of quadratic cost function $c(\tau) = \kappa \tau^2$, $\kappa > 0$. From (34), we have $\frac{\partial \hat{D}_r^i}{\partial r} \propto 3 - 2r$, which is negative when $r \geq 2$. Hence $\hat{D}_r^i$ is strictly decreasing in r for $r \geq 2$. We next show that $\hat{D}_2^i > \hat{D}_1^i$. Let $\bar{D}_r^i = \log(\hat{D}_r^i)$, $r = 1, 2$. We have

$$
\bar{D}_r^i = \log\left(\tau_V^i + \bar{\tau}_i^*(r)\right) - 2\rho\kappa r (\tau_i^*(r))^2,
$$

where $\tau_V^i = \frac{1}{\mathrm{Var}^{\widetilde{\tau}_\theta^i, \widetilde{\tau}_u^i}[\theta|p]}$, $\bar{\tau}_i^*(r) = (2 - 1/r)\tau_i^*(r)$. From (33), we have

$$
4\rho\kappa \bar{\tau}_i^*(r)(\tau_V^i + \bar{\tau}_i^*(r)) = \frac{1}{r}\left(2 - \frac{1}{r}\right)^2,
$$

so that

$$
\bar{\tau}_i^*(r) = \frac{1}{2}\left(-\tau_V^i + \sqrt{(\tau_V^i)^2 + \frac{\frac{1}{r}(2 - \frac{1}{r})^2}{\rho\kappa}} \right).
$$

Hence,

$$
\begin{aligned}
\bar{D}_r^i &= \log\left(\tau_V^i + \bar{\tau}_i^*(r)\right) - 2\rho\kappa r (\tau_i^*(r))^2 \\
&= \log\left(\tau_V^i + \bar{\tau}_i^*(r)\right) - \frac{2\rho\kappa r}{\left(2 - \frac{1}{r}\right)^2} (\bar{\tau}_i^*(r))^2 \\
&= \log\left(\frac{\tau_V^i + \sqrt{(\tau_V^i)^2 + \frac{\frac{1}{r}\left(2 - \frac{1}{r}\right)^2}{\rho\kappa}}}{2} \right) - \frac{2\rho\kappa r}{\left(2 - \frac{1}{r}\right)^2} \frac{2(\tau_V^i)^2 + \frac{\frac{1}{r}\left(2 - \frac{1}{r}\right)^2}{\rho\kappa} - 2\tau_V^i \sqrt{(\tau_V^i)^2 + \frac{\frac{1}{r}\left(2 - \frac{1}{r}\right)^2}{\rho\kappa}}}{4} \\
&= \log\left(1 + \sqrt{1 + \frac{\frac{1}{r}\left(2 - \frac{1}{r}\right)^2}{\rho\kappa(\tau_V^i)^2}} \right) - \log\left(2/\tau_V^i\right) - 1/2 \\
&\qquad - \frac{r}{2\left(2 - \frac{1}{r}\right)^2} \left(2\rho\kappa(\tau_V^i)^2 - 2\sqrt{(\rho\kappa(\tau_V^i)^2)^2 + \frac{1}{r}\left(2 - \frac{1}{r}\right)^2 \rho\kappa(\tau_V^i)^2} \right) \\
&=: q(r, b) - \log(2/\tau_V^i) - 1/2,
\end{aligned}
$$

where

$$q(r,b) = \log\left(1+\sqrt{1+\frac{\frac{1}{r}(2-\frac{1}{r})^2}{b}}\right) - \frac{1-\sqrt{1+\frac{\frac{1}{r}(2-\frac{1}{r})^2}{b}}}{\left(2-\frac{1}{r}\right)^2}br$$

with $b = \rho\kappa(\tau_V^i)^2$. As a result,

$$q(2,b) - q(1,b) = \log\left(\frac{1+\sqrt{1+\frac{9}{8b}}}{1+\sqrt{1+\frac{1}{b}}}\right) - \left(\frac{8}{9} - 1 - \frac{8}{9}\sqrt{1+\frac{9}{8b}} + \sqrt{1+\frac{1}{b}}\right)b.$$

We have

$$\begin{aligned}
&\frac{d(q(2,b)-q(1,b))}{db} \\
&= -\frac{\frac{1}{2}\frac{\frac{9}{8b^2}}{\sqrt{1+\frac{9}{8b}}}}{1+\sqrt{1+\frac{9}{8b}}} + \frac{\frac{1}{2}\frac{\frac{1}{b^2}}{\sqrt{1+\frac{1}{b}}}}{1+\sqrt{1+\frac{1}{b}}} + \frac{1}{9} + \frac{4}{9}\frac{2b+\frac{9}{8}}{\sqrt{b^2+\frac{9}{8}b}} - \frac{1}{2}\frac{2b+1}{\sqrt{b^2+b}} \\
&\propto \frac{8}{9}\frac{2b+\frac{9}{8}}{\sqrt{b^2+\frac{9}{8}b}} - \frac{9}{8}\frac{1}{b^2+\frac{9}{8}b+b\sqrt{b^2+\frac{9}{8}b}} + \frac{1}{b^2+b+b\sqrt{b^2+b}} - \frac{2b+1}{\sqrt{b^2+b}} + \frac{2}{9} \\
&= \frac{\frac{8}{9}(2b+\frac{9}{8})\left(b+\sqrt{b^2+\frac{9}{8}b}\right) - \frac{9}{8}}{b^2+\frac{9}{8}b+b\sqrt{b^2+\frac{9}{8}b}} + \frac{1-(2b+1)(b+\sqrt{b^2+b})}{b^2+b+b\sqrt{b^2+b}} + \frac{2}{9} \\
&\propto \frac{8}{9}\left(2b+\frac{9}{8}\right)\left(b+\sqrt{b^2+\frac{9}{8}b}\right)(\sqrt{b^2+b}+b+1) \\
&\quad - (2b+1)\left(b+\sqrt{b^2+b}\right)\left(\sqrt{b^2+\frac{9}{8}b}+b+\frac{9}{8}\right) - \frac{9}{8}\left(\sqrt{b^2+b}+b+1\right) \\
&\quad + \left(\sqrt{b^2+\frac{9}{8}b}+b+\frac{9}{8}\right) + \frac{2}{9}\sqrt{b^2+b}\left(b+\sqrt{b^2+b}\right)\left(\sqrt{b^2+\frac{9}{8}b}+b+\frac{9}{8}\right).
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{d(q(2,b)-q(1,b))}{db} \\
&\propto \frac{8}{9}\left(2b+\frac{9}{8}\right)\left(b+\sqrt{b^2+\frac{9}{8}b}\right)(\sqrt{b^2+b}+b+1) \\
&\quad - (2b+1)\left(b+\sqrt{b^2+b}\right)\left(\sqrt{b^2+\frac{9}{8}b}+b\right) - \frac{9}{4}(b+1)\left(\sqrt{b^2+b}+b\right) \\
&\quad + \sqrt{b^2+\frac{9}{8}b}+b+\frac{2b}{9}\left(\sqrt{b^2+\frac{9}{8}b}+b+\frac{9}{8}\right)\left(\sqrt{b^2+b}+b+1\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{16b}{9} + 1 - \frac{2b}{9}\left(b + \sqrt{b^2+b}\right) + 1 + \frac{2b}{9}\left(b + \sqrt{b^2+b}\right) + \frac{2b}{9}\right)\left(b + \sqrt{b^2 + \frac{9}{8}b}\right) \\
&\quad + \frac{b}{4}\left(\sqrt{b^2+b} + b + 1\right) - \frac{9}{4}(b+1)\left(\sqrt{b^2+b} + b\right) \\
&= 2(b+1)\left(b + \sqrt{b^2 + \frac{9}{8}b}\right) - \left(2b + \frac{9}{4}\right)\left(\sqrt{b^2+b} + b\right) + \frac{b}{4} \\
&= 2(b+1)\sqrt{b^2 + \frac{9}{8}b} - \left(2b + \frac{9}{4}\right)\sqrt{b^2+b} \\
&\propto (b+1) - \left(b + \frac{9}{8}\right) \\
&= -\frac{1}{8} < 0.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\lim_{b\to\infty}\left(q(2,b) - q(1,b)\right) &= \lim_{b\to\infty}\left(b\left(1 - \sqrt{1 + \frac{1}{b}}\right) - \frac{8b}{9}\left(1 - \sqrt{1 + \frac{9}{8b}}\right)\right) \\
&= \lim_{z\to 0}\left(\frac{1 - \sqrt{1+z}}{z} - \frac{8}{9}\frac{1 - \sqrt{1 + \frac{9z}{8}}}{z}\right) \\
&= -\frac{1}{2} + \frac{1}{2} = 0,
\end{aligned}
$$

and

$$
\lim_{b\to 0}\left(q(2,b) - q(1,b)\right) = \sqrt{\frac{9}{8}} > 0.
$$

Thus, $\hat{D}_2^i > \hat{D}_1^i$. The above arguments show that the optimal number of advisers in an equilibrium with endogenous quality of advisers is unique and $r_i^* = 2$ for every i. The proof is completed. $\square$

Proof of Proposition 7

By Proposition 6, any optimal number of advisers belongs to the interval $\left[\left\lfloor \frac{A_-+1}{2} \right\rfloor, \left\lceil \frac{A_++1}{2} \right\rceil\right] =: \mathcal{T}$. Given $r \in \mathcal{T}$. By $\lim_{\tau\to 0} c'(\tau) = 0$ and the strict convexity and twice continuous differentiability of $c(\cdot)$ (implying that $c'(\cdot)$ is increasing), we can see that there exists a unique positive root $\tau_i^*(r)$ to (33). Substituting $\tau_i^*(r)$ into (32), we get the maximum expected utility D_r^i when the number of advisers is r. Let r_i^* be the optimal number of advisers, i.e., $D_{r_i^*}^i = \max_{r\in\mathcal{T}} D_r^i$. If there are multiple optimal numbers of advisers, we choose the smallest one. Moreover, the equilibrium price is given by (6) with the replacement of $a = 1$, $\varphi = 0$, and $\Delta = \rho^{-1} \lim_{h\to\infty} \frac{\sum_{i=1}^{h} \tau_i^*}{h}$, where the limit exists under the regularity assumption given before this proposition. Moreover, it follows from Proposition 6 that the optimal number of advisers is two when the cost function is quadratic. The second part thus follows. The proof is completed. □

Proof of Proposition 8

Following the notations and the similar analysis in the proof of Proposition 1, from (16) we can get the following inequality

$$\begin{aligned} &-\frac{\rho\alpha}{\hat{\beta}\gamma}\varphi_i^2 + \frac{2\rho\alpha}{\gamma}(1-a_i)\varphi_i - \frac{\rho\alpha\hat{\beta}}{\gamma}(1-a_i)^2 \\ &\quad + \inf_{(\tau_j)_{j=1,\dots,n},\frac{1}{n}\sum_{j=1}^{n}\tau_j=\bar{\tau}_n} \sum_{j=1}^{n}(2a_{ij}-a_{ij}^2)\tau_j - \max_{1\le j\le n} \inf_{(\tau_s)_{s=1,\dots,n},\frac{1}{n}\sum_{s=1}^{n}\tau_s=\bar{\tau}_n} \tau_j > 0 \end{aligned}$$

From the above inequality, we can apply the similar arguments in the proof of Proposition 1 to show that $\sum_{j=1}^{n} a_{ij} = 1$ and $\varphi_i = 0$. □

Proof of Proposition 9

By virtue of Proposition 2, optimization problem (18) is equivalent to

$$\begin{aligned} \sup_{a_{ij},j=1,\dots,n} \inf_{\tau_j,j=1,\dots,n} \quad & \sum_{j=1}^{n}(2a_{ij}-a_{ij}^2)\tau_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} = 1, a_{ij} \ge 0, \\ & \frac{1}{n}\sum_{j=1}^{n}\tau_j = \bar{\tau}_n \end{aligned} \tag{35}$$

Fix $\{a_{ij}\}_{j=1,\dots,n}$ and consider optimization problem $\inf_{\tau_j,j=1,\dots,n} \sum_{j=1}^{n}(2a_{ij} - a_{ij}^2)\tau_j$. The lowest value of $(2a_{ij} - a_{ij}^2)$ will be given all the precision $n\bar{\tau}_n$ while other values receive zero precision. That is, $\tau_{j_1} = n\bar{\tau}_n$ for $j_1 \in \arg\min_{1\le j\le n}(2a_{ij} - a_{ij}^2)$, and $\tau_j = 0$ for $j \neq j_1$. Then the robust optimization problem (35) reduces to $\sup_{a_{ij},j=1,\dots,n} \min_{1\le j\le n} \ (2a_{ij} - a_{ij}^2)$, which clearly has solution $a_{ij} = 1/n$ for all j. The proof is completed. □

Proof of Proposition 10

With the change of the variable $\tau'_j = w_{ij}\tau_j$, the robust optimization problem (19) can be transferred into:

$$\begin{aligned} \sup_{a_{ij}, j=1,\dots,n} \inf_{\tau'_j, j=1,\dots,n} \quad & \sum_{j=1}^n \frac{2a_{ij} - a_{ij}^2}{w_{ij}} \tau'_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} = 1, a_{ij} \geq 0, \\ & \sum_{j=1}^n \tau'_j = K. \end{aligned} \tag{36}$$

Fix $\{a_{ij}\}_{j=1,\dots,n}$ and consider the following optimization problem:

$$\inf_{\tau'_j, j=1,\dots,n} \sum_{j=1}^n \frac{2a_{ij} - a_{ij}^2}{w_{ij}} \tau'_j.$$

The lowest value of $(2a_{ij} - a_{ij}^2)/w_{ij}$ will be given all the precision K while other values receive zero precision. That is, $\tau'_{j_1} = K$ for $j_1 \in \arg\min_{1\leq j\leq n}(2a_{ij} - a_{ij}^2)/w_{ij}$, and $\tau'_j = 0$ for $j \neq j_1$. Then, the robust optimization problem (36) reduces to $\sup_{a_{ij}, j=1,\dots,n} \min_{1\leq j\leq n} \ (2a_{ij} - a_{ij}^2)/w_{ij}$, which we next solve.

We claim that the optimal solution (still denoted as a_{ij}'s) must satisfy $a_{ij} > 0$ for any j and that

$$\frac{2a_{i1} - a_{i1}^2}{w_{i1}} = \frac{2a_{i2} - a_{i2}^2}{w_{i2}} = \cdots = \frac{2a_{in} - a_{in}^2}{w_{in}}. \tag{37}$$

Otherwise, we can increase a_{ij_1} by a small δ for $j_1 \in \arg\min_{1\leq j\leq n} \frac{2a_{ij} - a_{ij}^2}{w_{ij}}$ and decrease another a_{ij_2} with $\frac{2a_{ij_2} - a_{ij_2}^2}{w_{ij_2}} > \frac{2a_{ij_1} - a_{ij_1}^2}{w_{ij_1}}$ by δ to increase the lowest value of $\left\{\frac{2a_{ij} - a_{ij}^2}{w_{ij}}\right\}_{j=1,\dots,n}$.

Note that $w_{i1} \geq w_{i2} \geq \cdots \geq w_{in}$. It then follows from (37) that $a_{i1} \geq a_{i2} \geq \cdots \geq a_{in}$. From (37), we have

$$a_{ij} = 1 - \sqrt{1 + \frac{w_{ij}}{w_{i1}}(a_{i1}^2 - 2a_{i1})}, j = 2, \dots, n. \tag{38}$$

This is natural, recalling that a larger $w_{ij} > 0$ reflects greater relative confidence of investor i in the suggestions of adviser (i, j).

Finally, we can solve a_{i1} from the condition $\sum_{j=1}^n a_{ij} = 1$, i.e.,

$$\sum_{j=1}^n \left(1 - \sqrt{1 + \frac{w_{ij}}{w_{i1}}(a_{i1}^2 - 2a_{i1})}\right) = 1,$$

from which we can first determine a unique $0 < a_{i1} < 1$, and then $a_{ij}, j \geq 2$ by the equation (38). The proof is completed. □

Proof of Proposition 11

We first show that consultation increases Δ and then improves the price informativeness. From the expression of a_{ij}^* in Proposition 3 and (5), it suffices to show that

$$\sum_{j=1}^{n} a_{ij}^* \tau_j = \sum_{j=1}^{t} \left(1 - \frac{(t-1)\frac{\tau_t}{\tau_j}}{1 + \sum_{\ell=1}^{t-1} \frac{\tau_t}{\tau_\ell}} \right) \tau_j \geq \sum_{j=1}^{n} \tau_j / n,$$

which is equivalent to

$$(n-1)(\tau_1 + \tau_2 + \cdots + \tau_t) \geq \frac{(t-1)\tau_t}{\sum_{\ell=1}^{t} \frac{\tau_t}{\tau_\ell}} tn + \tau_{t+1} + \tau_{t+2} + \cdots + \tau_n.$$

The above is indeed true due to the relations

$$(\tau_1 + \tau_2 + \cdots + \tau_t) \sum_{\ell=1}^{t} \frac{\tau_t}{\tau_\ell} \geq t^2 \tau_t, \quad \tau_{t+1} + \tau_{t+2} + \cdots + \tau_n \leq (n-t)\tau_t,$$

and $\sum_{\ell=1}^{t} \frac{\tau_t}{\tau_\ell} \leq t$. The claim follows.

Second, recall that $\mathrm{Var}(\theta - p) = \left(\frac{\beta^2}{\tau_\theta} + \frac{1}{\tau_u} \right) / (\Delta + \beta)^2$, where $\beta = \frac{\tau_\theta}{\Delta \tau_u + \rho}$. Direct computations lead to

$$\begin{aligned}
\frac{\partial \,\mathrm{Var}(\theta - p)}{\partial \Delta} &= \frac{\frac{2\beta}{\tau_\theta} \frac{\partial \beta}{\partial \Delta} (\Delta + \beta)^2 - 2(\frac{\beta^2}{\tau_\theta} + \frac{1}{\tau_u})(\Delta + \beta)(1 + \frac{\partial \beta}{\partial \Delta})}{(\Delta + \beta)^4} \\
&= \frac{2}{(\Delta + \beta)^3} \left(\frac{\beta}{\tau_\theta} \frac{\partial \beta}{\partial \Delta} (\Delta + \beta) - \left(\frac{\beta^2}{\tau_\theta} + \frac{1}{\tau_u} \right) \left(1 + \frac{\partial \beta}{\partial \Delta} \right) \right) \\
&= \frac{2}{(\Delta + \beta)^3} \left(\frac{\beta}{\tau_\theta} \frac{\partial \beta}{\partial \Delta} \Delta - \frac{\beta^2}{\tau_\theta} - \frac{1}{\tau_u} - \frac{1}{\tau_u} \frac{\partial \beta}{\partial \Delta} \right) \\
&= \frac{2}{(\Delta + \beta)^3} \left(\left(\frac{\Delta}{\Delta \tau_u + \rho} - \frac{1}{\tau_u} \right) \frac{\partial \beta}{\partial \Delta} - \frac{1}{\tau_u} - \frac{\tau_\theta}{(\Delta \tau_u + \rho)^2} \right) \\
&= -\frac{2}{(\Delta + \beta)^3} \left(\frac{\Delta}{\Delta \tau_u + \rho} \frac{\tau_\theta \tau_u}{(\Delta \tau_u + \rho)^2} + \frac{1}{\tau_u} \right) \\
&< 0,
\end{aligned}$$

where we use the relation $\frac{\partial \beta}{\partial \Delta} = -\frac{\tau_\theta \tau_u}{(\Delta \tau_u + \rho)^2}$. Since we already know that consultation increases Δ, we can conclude that consultation decreases return volatility.

Third, direct calculations show that

$$\frac{\partial\left(\Delta+\frac{\tau_\theta}{\Delta\tau_u+\rho}\right)}{\partial\Delta}=1-\frac{\tau_\theta\tau_u}{(\Delta\tau_u+\rho)^2},$$

which is positive if Δ is large enough. Thus, we can conclude that consultation improves market liquidity in informationally efficient markets.

Finally, to show the last conclusion it suffices to consider the case where all the advisers in our economy have the same signal precision. From (5) and the definition of the benchmark economy, we know that consultation does not impact Δ, and then not p, and consequently improves equilibrium welfare by (10), (11) and the following relation

$$\sum_{j=1}^{n}(2a_{ij}^*-(a_{ij}^*)^2)\tau_j\geq\sum_{j=1}^{n}(2/n-1/n^2)\tau_j>\sum_{j=1}^{n}\tau_j/n.$$

The proof is completed. □

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