Strategic trading with uncertain market depth

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Abstract

We study a model of strategic informed traders submitting market orders together with noise traders where an uncertainty over the overall participation of strategic and noise traders leads to an uncertainty over market depth. Our analysis compares the main case with such uncertainty with the benchmark case without it. When liquidity is driven by informed trading (noise trading), expected trading volume is higher (lower) and expected price informativeness is lower (higher) in the main case compared with the benchmark case. We also analyze the effects of random variation of the aggregate participation, which confound the effects of market expansion and thereby possibly lead to higher expected trading volume and lower expected price informativeness following market expansion. Further, these results can explain a negative volume-volatility relation and a negative impact of transparency reforms on price informativeness.

Keywords: Market depth, liquidity, trading volume, price informativeness

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1 Introduction

The notion of liquidity, which broadly reflects a friction involved in trading for profits, plays an important role in explaining the observable and policy-relevant pattern of trading volume and price informativeness. Many previous studies analyze imperfect liquidity in financial markets in light of strategic trading (e.g., Lee and Kyle (2022), Lou and Rahi (2023), Rostek and Weretka (2012), Rostek and Weretka (2015)). They shed light on the relationship between market size and liquidity and provide testable predictions on trading volume and price informativeness, the latter of which can provide normative implications in various contexts of market expansion or fragmentation.

These established theoretical results draw on the premise that the participation of traders and the resulting market depth are common knowledge. It is inherent in most existing frameworks of strategic trading (e.g., Kyle (1985), Kyle (1989)). In these frameworks, strategic traders make a precise inference on the price impact of orders, which is endogenously pinned down by the participation of other traders and their strategic behavior. Despite an advantage of tractable analysis and the possibility of learning in the long run, this premise is not innocuous. For example, financial market participation is influenced by limited attention, which is unobservable, and institutional investors are not the exception (e.g., Ben-Raphael et al. (2017)). Indeed, such limited attention and the resulting uncertainty over market depth are consistent with Bali et al. (2014)'s finding that the stock market underreacts to stock-level liquidity shocks in the sense that these liquidity shocks predict future returns in the short run up to 6 months.

This paper presents a tractable analysis of strategic trading relaxing the usual premise of common knowledge on market depth. We argue that imperfect information on the aggregate market participation and the resulting Bayesian-rational inference on uncertain market depth may influence predictions regarding market size, trading volume and price informativeness in a systematic and qualitative manner.

In this regard, we study a market with strategic and noise traders who can buy or sell a risky asset. Strategic traders have private information about the asset. However, the market involves a randomness about the overall participation of strategic and noise traders so that strategic traders may be uncertain about that. Specifically, these strategic traders form beliefs over two possible states of nature: At the first state, only one strategic trader and one noise trader are randomly chosen to participate in the market, leading to low market depth. At the second state, all strategic and noise traders participate in the market, leading to higher market depth. Trade occurs via market orders (e.g., Kyle (1985)): Strategic and noise traders simultaneously submit market orders to a market-maker who observes the aggregate demand and then sets the price.

How does an uncertainty over market depth affect expected trading volume and price informativeness in equilibrium? To answer this question, we compare the main case with such uncertainty with the benchmark case where the overall participation of other traders is common knowledge after being realized. Key to our analysis is whether liquidity is driven by informed or noise trading, the former (latter) of which corresponds to a steeper (slower) growth of the number of participating strategic traders relative to that of noise variance as we move from the low-liquidity state to the high-liquidity one. In the benchmark case, when liquidity is driven by informed trading (noise trading), each strategic trader optimally chooses to be less (more) aggressive as we move from the low-liquidity state to the high-liquidity one. By contrast, in the main case, each trader without knowledge on the realized state chooses his trading aggressiveness to maximize his participation-conditional expectation of trading profits.

Our main results show that expected trading volume is higher (lower) and expected price informativeness is lower (higher) in the main case compared with those in the benchmark case when liquidity is driven by informed trading (noise trading). Intuitively, as we move from the benchmark case to the main case, conditional on the high-liquidity (low-liquidity) state, strategic traders tilt their aggressiveness toward the optimal one conditional on the low-liquidity (high-liquidity) state. This means that they trade more (less) aggressively when liquidity is driven by informed trading (noise trading). As a result, expected trading volume is higher (lower) conditional on the high-liquidity state and lower (higher) conditional on the low-liquidity state when liquidity is driven by informed trading (noise trading). The former change is always dominant in that it determines the sign of change in expected trading volume. On the other hand, price informativeness conditional on each state changes in the same direction. In contrast to trading volume, the latter change conditional on the low-liquidity state is always dominant.

These main results on expected trading volume and price informativeness shed light on two opposing economic forces that operate as we move from the benchmark case to the main case. By the first one arising from asymmetric price impacts across states and the resulting asymmetry of state-conditional profits, traders' aggressiveness is tilted toward the optimal one conditional on the low-liquidity state. By the second one arising from participation-conditional probabilities of states, traders' aggressiveness is tilted toward the optimal one conditional on the high-liquidity state. The opposite signs of changes in expected trading volume and price informativeness arise from their different weights on these opposite forces: The first one determines the sign of change in expected trading volume, whereas the second one determines the sign of change in expected price informativeness.

We conduct further analyses on the effects of market size and its state-wise variation. First, we investigate how expected trading volume and price informativeness change as the state-wise variation of aggregate participation increases keeping its average unchanged. Their changes are divided into those attributed to "randomness" (i.e., aggregate participation becoming state-dependent and known to strategic traders) and "uncertainty" (i.e., aggregate participation becoming unknown to informed traders), the latter of which corresponds to the above main results. The results show a similar insight: such state-wise variation increases (decreases) expected trading volume and decreases (increases) expected price informativeness when liquidity is predominantly driven by informed trading (noise trading).

Next, we use these results to confirm our motivating argument that the effects of market size on trading volume and price informativeness are confounded by a correlation between (average) market size and the state-wise variation of aggregate participation. With a sufficient number of strategic traders at the high-liquidity state, meaning that liquidity is driven by informed trading, the confounding effect via the state-wise variation of aggregate participation may lead to a negative effect of further increasing the number of strategic traders on expected price informativeness. This occurs with a contemporary increase in expected trading volume, which may be of higher order than trading volume with fixed market size in the limit of large number of strategic traders. More generally, we show that a variety of possible relations among market size, expected trading volume and expected price informativeness may occur. In particular, their relation hinges on how market size influences the numbers of strategic and noise traders at the high-liquidity state as well as the probability of the high-liquidity state.

Our main results provide further empirical and policy implications as follows. First, they suggest the possibility of large trading volume together with a weak relationship between trading volume and price volatility, the latter of which coincides with price informativeness in our model. These outcomes occur because a negative volume-volatility relation can be observed across different extents of uncertainty over market depth. At the same time, when liquidity is driven by informed trading, they explain qualitatively higher volume than predicted by the standard framework with fixed participation. Second, we discuss the effects of enhancing transparency of financial markets. Focusing on the side of transparency reforms providing information on the aggregate participation, our results suggest that such transparency reform may have an unintended consequence of reducing price informativeness when liquidity is driven by noise trading. It does so by resolving the uncertainty over market depth and thereby reversing the aforementioned force arising from participation-conditional probabilities of states.

The remainder of this paper is organized as follows. We review the related literature in Subsection 1.1. In Sections 2 and 3, we introduce and then solve the model of financial markets with uncertainty over market depth. Our main results are in Sections 4 and 5, and their further implications are in Section 6. We discuss possible extensions and conclude the paper in Section 7. The Appendix contains all proofs.

1.1 Related literature

The static modeling framework of strategic trading in this paper borrows from Kyle (1985) and its extension to multiple informed traders (e.g., Edmans and Manso (2011), Kyle and Wang (1997), Lambert et al. (2018)). The special case of fixed participation in our model (i.e., $q = 1$) would be a special case of the models in these previous studies.[1]

Uncertain market depth and/or numbers of participants have been considered in several previous studies in the theoretical literature. On the one hand, the dynamic framework of Kyle (1985) has been extended in this direction. Most of them considered time-varying noise variance

[1] Their models extend it in alternative directions, such as costly effort to improve firm value (Edmans and Manso (2011)), investor overconfidence (Kyle and Wang (1997)), and complex information structures (Lambert et al. (2018)).

whose stochastic process, which may involve time trends, is known to an informed trader (e.g., Collin-Dufresne and Fos (2016), Han et al. (2023)). Hong and Rady (2002) is closer to ours in that they analyzed the situation of its variance being stochastically persistent and initially unknown to informed traders. Without learning about its variance over time, the latter's framework would operate similarly to the special case of (fixed participation of) single strategic trader (i.e., $L_S = 1$ and $L_N \geq 2$) in our model.[2] Their focus is the possibility of learning from past prices and volume. On the other hand, Lauermann and Speit (2023) studied a common-value auction of a single indivisible good with uncertain number of participating bidders who are partially informed. The participating bidders infer the number of rivals from a Poisson prior and prices, resulting in the non-existence of equilibrium in large markets. Compared with these previous studies, our static modeling framework abstracts from the possibility of learning from prices.[3] Instead, the notion of market depth, which is specific to trading divisible assets, considered in this paper reflects the aggregate participation of strategic and noise traders, the former of which is particularly crucial for our motivating argument concerning market size formally analyzed in Subsection 5.2.

Our analysis contributes to the broad literature on the optimal structure and design of financial markets. Many previous studies investigated the effects of market expansion or fragmentation in various practical contexts such as risk sharing and decentralized exchanges (e.g., Kawakami (2017), Malamud and Rostek (2017), Chen and Duffie (2021)) and corporate governance (e.g., Edmans and Manso (2011)). The literature also suggests that non-obvious effects of changing market size on price informativeness can be obtained with heterogeneity in per-unit asset valuations (e.g., Rostek and Weretka (2012), Rostek and Weretka (2015), Lee and Kyle

[2] In fact, their setting involves *multiple* strategic traders who always participate in the market regardless of the variance of noise trade. It is not nested by the main case or the benchmark case of our model in Section 2 where one or $L_S \geq 2$ informed traders come with one or $L_N \geq 1$ noise traders, respectively. Nevertheless, their static equilibrium result does not qualitatively hinge on the (fixed) number of strategic traders. In particular, it is consistent with Proposition 5 regarding the negative effect of random and uncertain liquidity on individual trading volume (which is equivalent to expected trading volume in the main case of our model) when liquidity is driven by noise trading (i.e., $L_S < L_N^2$).

[3] It could be regarded as a reduced form of dynamic trading where informed traders are myopic and/or the (time-varying) state of nature is random in that it is not predictable from past prices and trading volume. We further discuss this point in Section 7.

(2022), Lou and Rahi (2023)) or that in trader characteristics (e.g., Kacperczyk et al. (2024)), as detailed in Subsection 5.2. Moreover, strategic trading is relevant for the design of decentralized exchanges, such as clearing (e.g., Rostek and Yoon (2021)). Our analysis abstracts from risk sharing and cross-exchange interaction to restrict attention to competition among strategic traders. Also, it assumes common per-unit value and common prior and information across strategic traders. Even with these conventional assumptions, our analysis highlights the role of state-wise variation of market size as a possible factor driving a complex relation among market size, trading volume and price informativeness, as detailed in Subsection 5.2. Such uncertainty of market size might be practically relevant in the context of market fragmentation, given the opaqueness of decentralized exchanges. Relatedly, as discussed in Section 6, the analysis provides implications concerning transparency on market size, which can be regarded as an element of exchange design. From a broad perspective, it is of interest that our model with "heterogeneity" across states can alternatively explain non-obvious outcomes which seemingly require heterogeneity across agents, given the analytical challenge of models with such agent-wise heterogeneity (e.g., Lambert et al. (2018), Kacperczyk et al. (2024)).

A line of previous studies examined the impact of transparency in financial markets. Gao and Liang (2013) investigated this question with a strategic-trading framework based on Kyle (1985) like ours, focusing on the side of transparency increasing asset-payoff-relevant information and shedding light on its potentially mixed impact on firm value via market liquidity and private information acquisition. Moreover, many other previous studies extended the standard REE model to highlight the possibility that such transparency has unintended consequences of reducing price informativeness via distorting information choices (e.g., Banerjee et al. (2018), Edmans et al. (2016)) and promoting noise-creating trades (e.g., Han et al. (2016)).[4] While they all focused on the side of transparency on asset-payoff-relevant information, this paper alternatively focuses on the side of transparency on market conditions (i.e., aggregate participation), showing its possibly negative effect on price informativeness. This side of transparency is also distinguished from learning about what others know, which has been addressed broadly (e.g., Morris and Shin (2002), Gao (2008)).

The presence of uncertainty over broadly defined market conditions is not uncommon in the

[4]See Kurlat and Veldkamp (2015) for a similar argument focusing more on the investor welfare.

literature. Building on Kyle (1985)'s framework of strategic trading like ours, Chakraborty and Yılmaz (2004) and Goldstein and Guembel (2008) analyzed markets with uncertainty over the presence and informedness of a single strategic trader, respectively. Their analyses generally shed light on the possibility of manipulation by the strategic trader, potentially reducing price informativeness. Also, Cipriani and Guarino (2008) and Park and Sabourian (2011) built on the framework of Glosten and Milgrom (1985), which involves an uncertainty over the informedness of trading counterparty, to find the possibility of informational cascades, where informed traders act independently of their own signal so that their private signals cannot be revealed. Beyond our focus on strategic trading, many previous studies built on REE models to introduce different types of uncertain market conditions and learning about them, explaining various equilibrium properties that would not be straightforward in the standard REE model (e.g., Banerjee and Green (2015), Gao et al. (2013), Papadimitriou (2023), Peress and Schmidt (2024)). Compared with the above studies on insider manipulation, informational cascades, and Gao et al. (2013)'s equilibrium multiplicity in a similar spirit, our analysis with market-depth uncertainty is relatively optimistic about the stability of markets, providing the existence and uniqueness of equilibrium outcome. Indeed, the two economic forces behind our main results (in Section 4) via asymmetric price impacts and participation-conditional probabilities of states do not arise in those existing frameworks.[5] Nevertheless, our finding on a possible decrease of price informativeness following market expansion (Subsection 5.2), which occurs with a contemporary increase in trading volume, is reminiscent of the aforementioned price-crash-like outcomes in the literature.

2 Model

There is a market to trade a security. The security has an asset value $\theta \in \mathbb{R}$, which is drawn from a normal distribution $\mathcal{N}(\theta_0, \sigma_0^2)$.

[5]For example, the aforementioned REE models rule out the state dependence of (individual) price impacts, and they assume that market participants hold an exogenous prior distribution on the proportion of informed traders rather than calculating their participation-conditional probabilities according to the Bayes' rule. Also, the framework of Glosten and Milgrom (1985) rules out strategic interaction across informed traders, which is essential in the former economic force.

The market consists of $L_S \geq 2$ strategic traders, $L_N \geq 1$ noise traders, and a competitive market-maker. The state of nature $a \in \{0, 1\}$ influences the participation of strategic and noise traders in the market. Specifically, if $a = 0$, one strategic trader and one noise trader are picked uniformly and randomly from the pool of L_S strategic traders and that of L_N noise traders, respectively, to participate in the market. If $a = 1$, all L_S strategic traders and L_N noise traders participate in the market. Denote by $q \in (0, 1)$ the prior probability that $a = 1$. The participation of each individual strategic trader i is denoted by $a_i \in \{0, 1\}$, where $a_i = 1$ represents the participation of the trader, and $a_i = 0$ otherwise: If $a = 1$, then $a_i = 1$ for every strategic trader i. If $a = 0$, then $a_i = 1$ for only one randomly chosen strategic trader i, and $a_i = 0$ for other strategic traders.

Strategic traders and noise traders who participate in the market submit market orders and the price is set by the market-maker. Each strategic trader i observes the asset value θ and submits an order to maximize his trading profit. By contrast, noise traders submit a common random order ω, which is drawn from normal distribution $\mathcal{N}(0, \sigma_\omega^2)$. The distribution of the asset value θ, the noise order ω, and the state of nature a are jointly independent.

For the sake of notational convenience, define $M_S = 1 + a\,(L_S - 1)$ and $M_N = 1 + a\,(L_N - 1)$ as the number of participating strategic traders and noise traders conditional on $a \in \{0, 1\}$, respectively. Also, denote by $\mathcal{I}_i$ the information set of strategic trader i detailed below. The timing is as follows: First, nature independently draws the asset value θ, the noise order ω, and the state of nature a. If $a = 0$, then $a_i = 1$ for only one randomly picked strategic trader, and $a_i = 0$ for others. If $a = 1$, then $a_i = 1$ for all strategic traders. Each strategic trader observes his information set $\mathcal{I}_i$ and noise traders observe ω. Second, conditional on $a_i = 1$, each (participating) strategic trader i submits a market order $x_i \in \mathbb{R}$ and each (participating) noise trader submits the order $\omega \in \mathbb{R}$. Then, the market-maker observes the aggregate demand $y = \sum_{i=1}^{L_S} x_i + M_N \omega$ and sets a price p. After the trade is made, the true asset value θ is realized and each strategic trader obtains a profit of $\pi_i(\theta, x_i, p) = x_i(\theta - p)$.

While the market-maker knows the state of nature a, we consider the following two cases regarding strategic traders' knowledge on it:

1. Benchmark case: Each strategic trader knows the asset value θ and the state of nature $a \in \{0, 1\}$. Including his own participation a_i, the information set is $\mathcal{I}_i = \{\theta, a, a_i\}$.

2. Main case: Each strategic trader i knows θ but does not know the state of nature $a \in \{0,1\}$. Including his own participation a_i, the information set is $\mathcal{I}_i = \{\theta, a_i\}$. Accordingly, he infers the state a conditional on $a_i = 1$, i.e.,

$$\Pr(a = 1|\mathcal{I}_i) = \frac{(L_S - 1)q}{(L_S - 1)q + 1} \frac{L_S}{L_S - 1}. \tag{1}$$

The posterior probability is always higher than the prior $q = \Pr(a = 1)$ and is increasing toward one as L_S increases toward infinity. These properties arise from "uniformly random picking", which implies that each strategic trader is less likely to be picked as the only one participating at the low-liquidity state $a = 0$ as the pool of traders is larger.

A demand function for strategic trader i is a mapping X_i from own participation a_i and information sets $\mathcal{I}_i$ to orders $X_i(a_i, \mathcal{I}_i)$. A pricing rule is a mapping P from states and aggregate demands to prices $P(a, y)$.

Definition 1. *An equilibrium is $((X_i^*)_{i=1}^{L_S}, P^*)$ that satisfies the following conditions:*

(i) For every i such that $a_i = 0$, $X_i^*(0, \mathcal{I}_i) = 0$.

(ii) For every i such that $a_i = 1$,

$$X_i^*(1, \mathcal{I}_i) = \arg\max_{x_i} \left[x_i(\theta - \mathbb{E}[P^*(a, y)|\mathcal{I}_i]) \right],$$

where i recognizes $y = x_i + \sum_{j \neq i} X_j^*(a_j, \mathcal{I}_j) + M_N \omega$, and his information set is $\mathcal{I}_i = \{\theta, a, a_i\}$ in the benchmark case, and $\mathcal{I}_i = \{\theta, a_i\}$ in the main case.

(iii) For each aggregate demand $y \in \mathbb{R}$,

$$P^*(a, y) = \mathbb{E}[\theta|a, y],$$

where the expectation is taken with respect to the distribution of aggregate demand $y = \sum_{i=1}^{L_S} X_i^*(a_i, \mathcal{I}_i) + M_N \omega$.

The first two conditions require that only participating strategic traders submit non-zero orders, and that each of them submits an order to maximize his own profit, taking into account that the asset value θ is mapped into others' orders, which translate to the distribution of the price $P^*(a, y)$ combined with his own order x_i. In the benchmark case, each realization of $a \in \{0, 1\}$ pins down the numbers of other strategic and noise traders participating in the market (i.e., M_S and M_N). Accordingly, for each a, the equilibrium definition in this case is identical to that in the previous framework of market orders with M_S strategic traders and the corresponding variance of noise trade (e.g., Kyle and Wang (1997), Lambert et al. (2018)).[6] In the main case, each strategic trader i makes an inference on the state of nature a, which would pin down M_S and M_N, conditional only on his own participation (i.e., $a_i = 1$). Such inference influences how the asset value θ is converted into the distribution of the price $P^*(a, y)$ given the trader's own order x_i described above.

In the third condition, the market-maker passively sets the competitive price given the aggregate demand $y = \sum_{i=1}^{L_S} X_i^*(a_i, \mathcal{I}_i) + M_N\omega$, rather than pursuing his own profit or objective. This assumption of passive market-making could be replaced by an explicit Bertrand auction among multiple risk-neutral bidders, each of whom observes the aggregate demand y (Kyle (1985)). At this point, the market-maker knows the number of strategic and noise traders participating in the market (i.e., $a \in \{0, 1\}$), which is not recognized by strategic traders who submit orders.

For the sake of tractability, our analysis restricts attention to linear equilibria defined as follows:

Definition 2. *Equilibrium* $((X_i^*)_{i=1}^{L_S}, P^*)$ *is linear if, for each* i, X_i^* *is a linear function of* θ *given the trader's information set* $\mathcal{I}_i$ *conditional on* $a_i = 1$*, and* P^* *is a linear function of aggregate demand* y *given* $a \in \{0, 1\}$*.*

2.1 Discussion of the assumptions

Among many other alternatives (e.g., Kyle (1989)), the current Kyle (1985)-based trading mechanism is employed to tractably capture our motivating situation with uncertain market depth in the short run. Specifically, our model extends a multi-trader variant of Kyle (1985), whose literature is reviewed in Subsection 1.1, to uncertainties over the numbers of participating strategic and noise traders. Regarding the latter uncertainty, the model is generally

[6] While a single noise trader exists in their models, increasing the number of noise traders in the current model is equivalent to (quadratically) increasing the variance of the single noise trader's trade in their models (i.e., setting it to be $M_N^2\sigma_\omega^2$).

consistent with Hong and Rady (2002), and its assumption of perfectly correlated noise orders is not critical.[7] Regarding the former uncertainty, the model operates in accordance with the standard (symmetric) Bayesian inference of participating strategic traders, including that on participation-conditional probabilities of states. As in most of the analyses in the existing literature, linearity is assumed in agents' equilibrium behavior (Definition 2).[8]

The benchmark case of the model is set to highlight our focus of analysis on the effects of uncertainty over market depth. As noted in Section 2, for $a = 0$ ($a = 1$), an equilibrium in the benchmark case corresponds to that of Kyle (1985) (its multi-trader variant), where the participation of traders is publicly known and pins down the market depth. By contrast, an equilibrium in the main case involves agents' behaviors in response to the aforementioned uncertainties over their participation. The difference of agents' behaviors between the benchmark case and the main case drives the analysis in Section 4.

The definition of equilibrium (Definition 1) draws on the premise that the market-maker has superior information about the numbers of participating strategic and noise traders (i.e., $a \in \{0, 1\}$) compared with strategic traders. This premise can be justified by empirical evidence on the behavior of market-makers responsive to market conditions (e.g., Anand and Venkataraman (2016)). Indeed, many of these market-makers are associated with investment banks, who have own research arms, and these affiliated market-makers appear to benefit from research coverage (e.g., Madureira and Underwood (2008)). Also, they might particularly benefit from such information on (short-term) market conditions as they are distinguished from typical traders in terms of trading frequency.

The assumption of strategic traders exogenously and perfectly knowing the asset value provides the clearest case of adverse selection between strategic traders and the market-maker.[9] As formally verified in Section 7, the main economic forces driving our main results in Section

[7] The number of participating noise traders in our model is captured by their state-dependent total variance considered in Hong and Rady (2002) without loss of generality. At this point, it is straightforward to see that our main results in Sections 4 and 5 continue to hold for the alternative case of $\hat{L}_N$ noise traders submitting *independent* orders by simply substituting $\hat{L}_N$ for L_N^2.

[8] See McLennan et al. (2017) for the uniqueness of such linear equilibrium in a broader class of equilibria focusing on the case of single strategic trader.

[9] Such adverse selection is indeed essential in the first economic force via asymmetric price impacts across states throughout Section 4.

4 generally persist even with noisy information on the asset value. Also, the possibility of endogenous information acquisition is discussed in Section 7.

The form of randomness in the numbers of participating strategic and noise traders given by M_S and M_N, respectively, is stylized in the current model for the sake of simplicity. The positive state-wise correlation between the numbers of participating strategic and noise traders is driven by our initial motivation concerning the (state-wise) variation of market depth. Also, there is no loss of generality regarding the number of participating noise traders except for being binary.[10] Still, the model imposes a restriction on the number of participating strategic traders M_S in that it can be either one or L_S. One possible interpretation is that strategic traders do not distinguish whether their own attentiveness is "private" (i.e., the trader is aware of the relevance of that news alone) and "public" (i.e., all traders are aware).[11] In addition, as seen in Section 4 and more generally discussed in Section 7, the core intuition behind our main results does not critically depend on the restriction that $M_S = 1$ in state 0.

3 Equilibrium characterization

In this section, we establish the existence of a unique equilibrium of the game in the benchmark case and the main case.

For each state $a \in \{0, 1\}$, an equilibrium is determined by the following "*guess and check*" procedures: First, we conjecture that each participating strategic trader i's order is $X_i(1, \mathcal{I}_i) = \beta(\theta - \theta_0)$, where $\beta \in \mathbb{R}^+$. Then the market-maker sets the price

$$P(y) = \mathbb{E}\left[\theta | a, y\right] = \theta_0 + \lambda y,$$

where λ is given by the Projection Theorem for normal random variables as follows:

$$\lambda = \frac{1}{M_S \beta} \frac{\sigma_0^2}{\sigma_0^2 + \frac{M_N^2 \sigma_\omega^2}{M_S^2 \beta^2}}. \tag{2}$$

[10] Specifically, the situation where $\underline{L}_N$ ($\overline{L}_N$) noise traders participate in state 0 (1) is equivalent to the model where $L_N = \overline{L}_N / \underline{L}_N$ and each noise trader's variance is multiplied by $\underline{L}_N^2$.

[11] This type of dichotomy between private and public news is broadly used as a simplifying assumption in the theoretical literature (e.g., Morris and Shin (2002)).

The equilibrium value of λ is often called Kyle's lambda (Kyle (1985)) and represents the degree of illiquidity. Second, each strategic trader i correctly recognizes the equilibrium price as a function of his order x_i, others' orders $x_{-i} := \sum_{j\in\{1,\cdots,L_S\}\setminus\{i\}} x_j$ and the noise order $M_N\omega$, i.e., $P_i : \mathbb{R}^3 \to \mathbb{R}$ such that

$$P_i\left(x_i, \theta, M_N\omega\right) = \theta_0 + \lambda\left[x_i + (M_S - 1)\beta(\theta - \theta_0) + M_N\omega\right], \tag{3}$$

combined with the fact that $x_{-i} = (M_S-1)\beta(\theta-\theta_0)$ since $x_j = \beta(\theta-\theta_0)$ for each participating strategic trader $j \neq i$ by the initial conjecture. Given $a_i = 1$ and θ, he chooses the order $x_i = X_i(1, \mathcal{I}_i)$ so as to maximize his expected profit

$$\mathbb{E}\left[x_i\left(\theta - P_i(x_i, \theta, M_N\omega)\right)|\mathcal{I}_i\right],$$

where P_i is obtained from Equation (3). Noting that the expected profit is hump-shaped with respect to x_i, its first-order condition is

$$\begin{aligned}\frac{d\mathbb{E}\left[x_i\left(\theta - P_i(x_i, \theta, M_N\omega)\right)|\mathcal{I}_i\right]}{dx_i} &= \mathbb{E}[\theta - P_i(x_i, \theta, M_N\omega)|\mathcal{I}_i] - \mathbb{E}[\lambda|\mathcal{I}_i]x_i \\ &= \theta - \theta_0 - \mathbb{E}[\lambda(M_S-1)|\mathcal{I}_i]\beta(\theta-\theta_0) - 2\mathbb{E}[\lambda|\mathcal{I}_i]x_i = 0, \end{aligned} \tag{4}$$

where λ is given by Equation (2). The first term $\mathbb{E}[\lambda(M_S-1)|\mathcal{I}_i]$ represents trader i's inference about the covariance between the price and the asset value θ given conjectured trading coefficient β. Other things being equal, it negatively influences his trading incentive. The second term $\mathbb{E}[\lambda|\mathcal{I}_i]$ represents trader i's inference about the price impact, which negatively influences his trading incentive as well.

3.1 Equilibrium in the benchmark case

In the benchmark case, each strategic trader knows the realized state a, which pins down the numbers of participating strategic and noise traders (i.e., M_S and M_N, respectively) and thus λ through Equation (2). Consequently, they correctly recognize the true values of expectation terms in Equation (4), i.e., $\mathbb{E}[\lambda(M_S-1)|\mathcal{I}_i] = \lambda(M_S-1)$ and $\mathbb{E}[\lambda|\mathcal{I}_i] = \lambda$. Then Equation (4) is solved to obtain the trader's optimal order and then check whether it is consistent with the initial conjecture β, leading to the following proposition:

Proposition 1. *In the benchmark case, there exists a unique linear equilibrium. In the unique linear equilibrium,*

(i) Kyle's lambda is $\lambda_b^* = \frac{\sqrt{M_S}}{M_S+1}\frac{1}{M_N}\sqrt{\frac{\sigma_0^2}{\sigma_\omega^2}}$*, and it is lower at* $a = 1$ *compared with* $a = 0$ *for every* $L_S \geq 2$ *and* $L_N \geq 1$*.*

(ii) Strategic traders' orders take the form of $X_i^*(0, \mathcal{I}_i) = 0$ *and* $X_i^*(1, \mathcal{I}_i) = \beta_b^*(\theta - \theta_0)$*, where*

$$\beta_b^* = \sqrt{\frac{M_N^2 \sigma_\omega^2}{M_S \sigma_0^2}} = \sqrt{\frac{(1 + a(L_N - 1))^2 \sigma_\omega^2}{(1 + a(L_S - 1))\sigma_0^2}}.$$

In particular, β_b^* *is lower (higher) at* $a = 1$ *compared with* $a = 0$ *if* $L_S > L_N^2$ $(L_S < L_N^2)$*.*

As the state of nature moves from $a = 0$ to $a = 1$, strategic traders face smaller market power or, equivalently, higher market depth measured by Kyle's lambda. The increase in market depth is attributed to both increases in the number of participating strategic traders M_S and that of participating noise traders M_N. This is consistent with Kyle (1985)'s notion of market depth, which is proportional to noise trade and inversely proportional to the amount of private information not incorporated in the price. Intuitively, an increase in M_N alleviates adverse selection given the information gap between strategic traders and the market-maker. Also, an increase in M_S reduces their information gap by making the price more informative, thereby reducing adverse selection given noise trade.

On the other hand, as the state of nature moves from $a = 0$ to $a = 1$, each strategic trader's aggressiveness β_b^* changes via two possibly opposing effects. First, as seen above, strategic traders face higher market depth and thus choose to trade more aggressively other things being equal. Second, they face smaller trading opportunities in equilibrium unless L_N is particularly large compared with L_S.[12] The comparison between these effects determines whether each strategic trader's aggressiveness β_b^* is lower or higher at $a = 1$ compared with $a = 0$. Specifically, if higher liquidity at $a = 1$ is mainly driven by informed trading (i.e., $L_S > L_N^2$), the second effect of smaller trading opportunities is dominant so that trading aggressiveness β_b^* is lower at $a = 1$. On the other hand, if it is mainly driven by noise trading (i.e., $L_S < L_N^2$), the first effect of higher market depth is dominant so that trading aggressiveness β_b^* is higher at $a = 1$.

[12]Formally, each strategic trader's equilibrium profit is $\mathbb{E}[\pi_b^*] = \mathbb{E}[\beta_b^*(\theta - \theta_0)(\theta - p)] = \frac{M_N}{(M_S+1)\sqrt{M_S}}\sqrt{\sigma_0^2\sigma_\omega^2}$, which is lower at $a = 1$ compared with $a = 0$ if and only if $L_N < \frac{1}{2}(L_S + 1)\sqrt{L_S}$.

3.2 Equilibrium in the main case

In the main case, the two expectation terms $\mathbb{E}[\lambda(M_S-1)|\mathcal{I}_i]$ and $\mathbb{E}[\lambda|\mathcal{I}_i]$ in Equation (4) no longer equal their respective true values due to the lack of information about a. Instead, they are participation-conditional-probability-weighted average of the involved variables, where the posterior probability is given by (1). These expectation terms pin down state-invariant trading aggressiveness β_m^* as shown by the following proposition:

PROPOSITION **2.** *In the main case, there exists a unique linear equilibrium. In the unique linear equilibrium, strategic traders' orders take the form of $X_i^*(0,\mathcal{I}_i)=0$ and $X_i^*(1,\mathcal{I}_i)=\beta_m^*(\theta-\theta_0)$, where β_m^* is determined by*

$$\frac{\sigma_\omega^2}{\sigma_0^2(\beta_m^*)^2}=\frac{L_S^2}{2L_N^2}\left(\frac{\frac{L_N^2}{L_S^2}(1-q-L_Sq)+2q-1}{(L_S-1)q+1}+\sqrt{\left(\frac{\frac{L_N^2}{L_S^2}(1-q-L_Sq)+2q-1}{(L_S-1)q+1}\right)^2+\frac{4\frac{L_N^2}{L_S^2}}{(L_S-1)q+1}}\right).$$

Their individual aggressiveness β_m^* is compared with that in the benchmark case as follows:

(i) When liquidity is driven by informed trading (i.e., $L_S>L_N^2$), $\beta_b^*|_{a=1}<\beta_m^*<\beta_b^*|_{a=0}$.

(ii) When liquidity is driven by noise trading (i.e., $L_S<L_N^2$), $\beta_b^*|_{a=0}<\beta_m^*<\beta_b^*|_{a=1}$.

The second part of Proposition 2 presents the relation between β_m^* with β_b^* for $a\in\{0,1\}$. Regardless of whether $L_S>L_N^2$ or $L_S<L_N^2$, each trader's main-case aggressiveness is in the middle of those in the benchmark case conditional on two possible states. This property naturally follows from the fact that each trader maximizes a participation-conditional-probability-weighted average of profits for the two states 0 and 1, both of which have their own maximum at $\beta_b^*|_{a=0}$ and $\beta_b^*|_{a=1}$, respectively. Only under the knife-edge condition $L_S=L_N^2$, the optimal aggressiveness of each trader in the benchmark case is the same across the two states, so it holds $\beta_m^*=\beta_b^*|_{a=1}=\beta_b^*|_{a=0}$.

As a precursor to the main analysis in Section 4, we consider three different scenarios of large number of strategic investors (i.e., large L_S):

Scenario 1. High liquidity driven by informed trading: $L_S\to\infty$ and L_N is fixed;

Scenario 2. High liquidity driven by noise trading: $L_S\to\infty$ and $\frac{L_N^2}{L_S}\to\infty$;

Scenario 3. "Middle" scenario: $L_S \to \infty$ and $\frac{L_N^2}{L_S} \to \rho$ for some $\rho \in (0, \infty)$.

The following lemma shows that a closed-form representation of equilibrium is feasible for each of these large-market scenarios.

LEMMA **1.** *The following statements hold in the three scenarios of large markets:*

(i) In the first large-market scenario, $\beta_m^* \to \sqrt{\frac{\sigma_\omega^2(1-2q)}{\sigma_0^2}}$ *if* $q < \frac{1}{2}$, *and* $\sqrt{L_S}\beta_m^* \to L_N\sqrt{\frac{q}{2q-1}\frac{\sigma_\omega^2}{\sigma_0^2}}$ *if* $q > \frac{1}{2}$.

(ii) In the second large-market scenario, $\frac{\sqrt{L_S}}{L_N}\beta_m^* \to \sqrt{\frac{\sigma_\omega^2 q}{\sigma_0^2}}$.

(iii) In the third large-market scenario, $\beta_m^* \to \sqrt{\frac{\sigma_\omega^2}{\sigma_0^2\left[\frac{-(\rho-2)q-1+\sqrt{((\rho-2)q+1)^2+4\rho q}}{2\rho q}\right]}}$.

To get the intuition behind the lemma, we first note that each strategic trader's aggressiveness β_m^* in the main case maximizes the probability-weighted sum of profits conditional on $a = 0$ and $a = 1$, whose weight is given by Equation (1). Likewise, its first-order condition can be represented as the probability-weighted sum of first-order derivatives of profits conditional on $a = 0$ and $a = 1$. Formally, given conjectured coefficient of other traders β, Equation (4) is equivalent to

$$\Pr(a=1|\mathcal{I}_i)\left[\underbrace{\theta-\theta_0-\lambda|_{a=1}(L_S-1)\beta(\theta-\theta_0)}_{\text{Trading opportunities at } a=1}-\underbrace{2\lambda|_{a=1}}_{\text{Market power at } a=1}x_i\right]$$
$$+(1-\Pr(a=1|\mathcal{I}_i))\left[\underbrace{\theta-\theta_0}_{\text{Trading opportunities at } a=0}-\underbrace{2\lambda|_{a=0}}_{\text{Market power at } a=0}x_i\right]=0,$$

where $\Pr(a = 1|\mathcal{I}_i)$ is given by Equation (1) and $\lambda|_{a=0}$ and $\lambda|_{a=1}$ represent the (β-conditional) slopes of the pricing rule given by Equation (2) conditional on $a = 0$ and $a = 1$, respectively.[13]

Which state of nature $a \in \{0, 1\}$ is dominant over the other in determining his main-case trading aggressiveness β_m^*? Two opposing effects are present in each of the three scenarios of large markets. First, his marginal incentive to trade is flatter with respect to x_i conditional on

[13]As seen by the fact that Equation (2) is common to the benchmark case and the main case, these slopes are not different from the corresponding ones in the benchmark case, given that they are chosen by the market-maker, who knows the realized state, given β.

$a = 1$ compared with that conditional on $a = 0$. The difference of these slopes stems from the fact that negligible price impact leads to smaller profit loss from deviating from the optimal aggressiveness conditional on $a = 1$ compared with that conditional on $a = 0$. As a result, his aggressiveness β_m^* is tilted toward the optimal aggressiveness conditional on $a = 0$. Second, each strategic trader puts a negligible probability weight on $a = 0$ via the Bayesian inference conditional on his participation as noted in Section 2. This effect tilts his trading aggressiveness toward that conditional on $a = 1$.

In the first large-market scenario, his main-case trading aggressiveness β_m^* is predominantly driven by the first (second) effect if $q < \frac{1}{2}$ ($q > \frac{1}{2}$). Intuitively, as the prior probability q is lower so that the true state is more likely to be $a = 0$, the first effect, which pushes β_m^* toward the optimal aggressiveness conditional on $a = 0$ given the posterior probability $\Pr(a = 1|\mathcal{I}_i) \in (0, 1)$, tends to be stronger. At the same time, the second effect, which pushes β_m^* toward the optimal aggressiveness conditional on $a = 1$ (i.e., zero by Proposition 1) via an increase in the posterior probability $\Pr(a = 1|\mathcal{I}_i) \in (0, 1)$, tends to be weaker. The discontinuity at $q = \frac{1}{2}$ arises from the fact that these two opposite effects grow together in the limit of large L_S, eventually resulting in one effect dominating the other.

In the second large-market scenario, his main-case trading aggressiveness β_m^* is predominantly driven by the second effect provided that $q > 0$. In contrast to the corresponding (second) effect in the first large-market scenario, which decreases β_m^* toward zero, the second effect in this scenario increases β_m^* toward infinity as the optimal aggressiveness conditional on $a = 1$ goes to infinity.[14] Such diverging effect is likely to be stronger compared with the corresponding (second) effect in the first large-market scenario. On the other hand, the first effect from the difference of slopes across states is *not* particularly stronger in this scenario compared with the corresponding (first) effect in the first large-market scenario.[15]

In the third large-market scenario, the two effects balance each other in that β_m^* converges toward a positive number. Such balance occurs as both effects send β_m^* toward positive numbers

[14] It follows from Proposition 1 and $L_N^2/L_S \to \infty$ in this scenario.

[15] Specifically, in Equation (4), $\lambda|_{a=1}$ and $\lambda|_{a=0}$ as functions of β do not differ between the first and second large-market scenarios, as easily seen by Equation (2). Intuitively, given conjectured trading coefficient β, the former goes to zero due to large informed trading in both scenarios, whereas the latter does not differ between the two scenarios given the fixed numbers of participating strategic and noise traders (i.e., one for each type).

rather than sending it toward zero or diverging. As expected, the ratio of balance between the two effects in determining β^*_m hinges on $\rho\,(= L_N^2/L_S)$ as well as q. As implied by Proposition 2, β^*_m is between $\beta^*_b|_{a=0} = \sqrt{\frac{\sigma^2_\omega}{\sigma^2_0}}$ and $\beta^*_b|_{a=1} = \sqrt{\frac{\rho\sigma^2_\omega}{\sigma^2_0}}$.[16]

Remark. The two opposing effects described above are not specific to the three scenarios of large markets. Rather, they are present in deviating the main-case trading aggressiveness β^*_m from the prior-weighted average of the benchmark-case trading aggressiveness β^*_b for every $L_S \geq 2$ and $L_N \geq 1$.[17] The relative strength between these effects is determined by L_S and L_N in line with the three large-market scenarios presented above. As is clear in Section 4, these effects are also useful to explain changes in expected trading volume and price informativeness in response to a change from the benchmark case to the main case.

4 Trading volume and price informativeness

In this section, we compare expected trading volume and price informativeness across the benchmark case and the main case. Throughout the section, we restrict attention to the case where state a is relevant for each trader's aggressiveness (i.e., $L_S > L_N^2$ or $L_S < L_N^2$). Otherwise (i.e., $L_S = L_N^2$), trading volume and price informativeness would not differ across the benchmark case and the main case.

4.1 Effect of market-depth uncertainty on expected trading volume

Given that trading volume is readily observable, an analysis of expected trading volume, which refers to the prior-weighted expectation of trading volume as formally defined below, provides testable implications.

Definition 3. *Expected trading volume is defined as the sum of the expected absolute values of strategic and noise traders' equilibrium orders as follows:*

$$TV = \sum_{i=1}^{L_S} \mathbb{E}[|X^*_i(a_i, \mathcal{I}_i)|] + \mathbb{E}[M_N|\omega|],$$

[16] It follows from Proposition 1 and $L_N^2/L_S \to \rho$ in this scenario.

[17] The first effect persists because $\lambda|_{a=0} > \lambda|_{a=1}$ holds generically. Likewise, the second effect persists because the posterior probability $\Pr(a = 1|\mathcal{I}_i)$ is generally higher than the prior $q = \Pr(a = 1)$.

where the expectation is taken with respect to the state $a \in \{0,1\}$, the asset value θ, and the noise order ω. Denote by TV_b and TV_m the expected trading volume for the benchmark case and the main case, respectively.

In general, expected trading volume can be decomposed into the (expected) volume of informed trading and that of noise trading, given the independence between the asset value θ and the noise order ω:[18]

$$TV = \underbrace{\mathbb{E}[M_S\beta^*]\sqrt{\frac{2}{\pi}\sigma_0^2}}_{\text{Informed trading}} + \underbrace{\mathbb{E}[M_N]\sqrt{\frac{2}{\pi}\sigma_\omega^2}}_{\text{Noise trading}}, \tag{5}$$

where $\beta^* = \beta_b^*$ for the benchmark case, and $\beta^* = \beta_m^*$ for the main case.

In the benchmark case, we apply Equation (5) for $\beta^* = \beta_b^*$ and Proposition 1 to obtain

$$TV_b = \sqrt{\frac{2}{\pi}\sigma_\omega^2}\left[\underbrace{\mathbb{E}[M_N\sqrt{M_S}]}_{\text{Informed trading}} + \underbrace{\mathbb{E}[M_N]}_{\text{Noise trading}}\right], \tag{6}$$

which is consistent with Proposition 1 in light of its inverse relation with Kyle's lambda λ^*.

As we move from the benchmark case to the main case, each strategic trader's aggressiveness changes from β_b^* to β_m^* for each state $a \in \{0,1\}$ so that the volume of informed trading in Equation (5) changes from Equation (6) to

$$TV_m = \underbrace{\mathbb{E}[M_S]\beta_m^*\sqrt{\frac{2}{\pi}\sigma_0^2}}_{\text{Informed trading}} + \underbrace{\mathbb{E}[M_N]\sqrt{\frac{2}{\pi}\sigma_\omega^2}}_{\text{Noise trading}}. \tag{7}$$

More specifically, strategic traders respond to the absence of information about state a depending on L_S and L_N and their aggregate trading volume changes accordingly as follows:

1. When liquidity is driven by informed trading (i.e., $L_S > L_N^2$), strategic traders are optimally more (less) aggressive in the main case compared with the benchmark case conditional on the state of high liquidity $a = 1$ (low liquidity $a = 0$). Formally, Proposition 2 implies $\beta_m^* > \beta_b^*$ ($\beta_m^* < \beta_b^*$) conditional on $a = 1$ ($a = 0$). As a result, conditional on $a = 1$ ($a = 0$), trading volume is higher (lower) in the main case compared with the benchmark case.

[18] The following equation immediately follows from Definition 3 together with the fact that the expectation of $|X|$ is equal to $\sqrt{\frac{2}{\pi}\sigma_X^2}$ for normal random variable X with mean zero and variance σ_X^2.

2. When liquidity is driven by noise trading (i.e., $L_S < L_N^2$), strategic traders are optimally less (more) aggressive in the main case compared with the benchmark case conditional on the state of high liquidity $a = 1$ (low liquidity $a = 0$). Formally, Proposition 2 implies $\beta_m^* < \beta_b^*$ ($\beta_m^* > \beta_b^*$) conditional on $a = 1$ ($a = 0$). As a result, conditional on $a = 1$ ($a = 0$), trading volume is lower (higher) in the main case compared with the benchmark case.

Whether liquidity is driven by informed or noise trading comes down to the comparison between the change of expected trading volume conditional on $a = 0$ and that conditional on $a = 1$. The following proposition shows that the change conditional on $a = 1$ is unambiguously dominant in determining the sign of change in expected trading volume.

Proposition **3.** *For every $L_S \geq 4$ and $L_N \geq 4$, the following statements hold:*

(i) When liquidity is driven by informed trading (i.e., $L_S > L_N^2$), expected trading volume in the main case is higher than that in the benchmark case.

(ii) When liquidity is driven by noise trading (i.e., $L_S < L_N^2$), expected trading volume in the main case is lower than that in the benchmark case.

The sign of change in expected trading volume (i.e., $TV_m - TV_b$) is pinned down by whether liquidity is driven by informed trading ($L_S > L_N^2$) or noise trading ($L_S < L_N^2$). While the proposition holds for $L_S \geq 4$ and $L_N \geq 4$, our simulations show that the robustness results in Proposition 3 also hold when $2 \leq L_S \leq 3$ and/or $1 \leq L_N \leq 3$.[19]

In order to see the intuition in detail, note first that trading volume in equilibrium is determined so as to ensure that each trader has no incentive to be further aggressive. Formally, conjecturing that every trader's aggressiveness is β, applying Equation (2) to Equation (4) yields

$$\left(1 - \mathbb{E}\left[\frac{\sigma_0^2}{\sigma_0^2 + \frac{M_N^2\sigma_\omega^2}{\nu_a^2}} \frac{M_S + 1}{M_S} \middle| \mathcal{I}_i\right]\right)(\theta - \theta_0) = 0, \tag{8}$$

[19]Specifically, the proof of the proposition defines a polynomial g as a function of q where $TV_m > TV_b$ if and only if $g(q) > 0$, regardless of other parameters in the model. For $2 \leq L_S \leq 3$ and/or $1 \leq L_N \leq 3$, the simulations show $g(q) > 0$ for all numerical values of $q \in [0, 1]$ considered.

where the expectation is taken over $a \in \{0,1\}$ and $\nu_a := M_S \beta$. As seen in Equation (5), the difference in trading volumes between the main case and the benchmark case comes from that in ν_a for each $a \in \{0,1\}$. Denote by $I_a = I_a(\nu_a)$ the left-hand side of Equation (8) conditional on each a normalized over $\theta - \theta_0$, i.e.,

$$I_a(\nu_a) := 1 - \frac{\sigma_0^2}{\sigma_0^2 + \frac{M_N^2 \sigma_\omega^2}{\nu_a^2}} \frac{M_S + 1}{M_S},$$

where $M_S = 1 + a(L_S - 1)$ and $M_N = 1 + a(L_N - 1)$. It is intuitive that $I_a(\nu_a)$ is decreasing in ν_a for each a because more informed trading leads to more informative price, which means less trading opportunity for each trader despite lower market power. In the benchmark case, the expectation in Equation (8) is equal to the true value, implying that ν_a is determined for each a, i.e., $I_0(\nu_0) = I_1(\nu_1) = 0$. In the main case, Equation (8) can be written as

$$\Pr(a = 1|\mathcal{I}_i) I_1(\nu_1) + \Pr(a = 0|\mathcal{I}_i) I_0(\nu_0) = 0,$$

which pins down $\nu_1 = L_S \beta_m^*$ and $\nu_0 = \beta_m^*$ so that $\nu_1 = L_S \nu_0$ since β_m^* is invariant to the state.

When liquidity is driven by informed trading (i.e., $L_S > L_N^2$), ν_1 increases and ν_0 decreases as we move from the benchmark case to the main case.[20] The former increase in ν_1 (latter decrease in ν_0) is equivalent to the increase (decrease) in trading volume conditional on $a = 1$ ($a = 0$) described above. The proposition tells us that, as we move from the benchmark case to the main case, $\mathbb{E}[\nu_a]$ increases due to the dominance of the increase in ν_1. Here, there are two opposing economic forces at work: (i) I_1 is decreasing in ν_1 with a flatter slope than I_0 does so in ν_0 due to lower price impact at $a = 1$ compared with $a = 0$. This implies that, if the probability weights were prior ones (i.e., $\Pr(a = 1|\mathcal{I}_i) = q$), the (prior-weighted) average $\mathbb{E}[\nu_a]$ would be higher in the main case compared with that in the benchmark case.[21] This effect moves the change in $\mathbb{E}[\nu_a]$ in the positive direction in response to a change from the benchmark case to the main case. In the limit of large L_S, it coincides with the first effect from

[20]Proposition 1 implies $\nu_1/L_S < \nu_0$ in the benchmark case since the left-hand side is $\beta_b^*|_{a=1}$, whereas the right-hand side is $\beta_b^*|_{a=0}$. Accordingly, ν_1 must increase and ν_0 must decrease so that $I_1(\nu_1) < 0$ and $I_0(\nu_0) > 0$ in the main case, where $\nu_1/L_S = \nu_0$ must hold.

[21]Graphically, as we move from the benchmark case to the main case, ν_1 (ν_0) increases (decreases) along I_1 (I_0) to make a balance between $I_1(\nu_1) < 0$ and $I_0(\nu_0) > 0$ so that $qI_1(\nu_1) + (1-q)I_0(\nu_0) = 0$. At this point, the flatter (steeper) slope of I_1 (I_0) means that ν_1 (ν_0) must increase further (decrease less).

asymmetric slopes of the marginal incentive of each trader described in the intuition behind Lemma 1. (ii) As the probability weights are actually tilted toward $a = 1$ compared with prior ones (i.e., $\Pr(a = 1|\mathcal{I}_i) > q$), the decrease in ν_0 is larger and the increase in ν_1 is smaller. This effect moves the change in $\mathbb{E}[\nu_a]$ in the negative direction in response to a change from the benchmark case to the main case. In the limit of large L_S, it coincides with the second effect from participation-conditional probabilities of states described in the intuition behind Lemma 1. According to the proposition, the first force (i) is dominant, leading to the dominance of increase in trading volume at $a = 1$.

When liquidity is driven by noise trading (i.e., $L_S < L_N^2$), ν_1 decreases and ν_0 increases as we move from the benchmark case to the main case.[22] The former decrease in ν_1 (latter increase in ν_0) is equivalent to the decrease (increase) in trading volume conditional on $a = 1$ ($a = 0$) described above. As in the above case of liquidity driven by informed trading, the proposition indicates that, as we move from the benchmark case to the main case, $\mathbb{E}[\nu_a]$ decreases due to the dominance of the decrease in ν_1, which stems from the dominance of the first economic force from asymmetric price impacts.

The dominance of change in trading volume conditional on $a = 1$ (i.e., ν_1) can be understood more intuitively as follows. We think of each trader who faces an uncertainty over the realized state a. If he knew it, his optimal aggressiveness would be dependent on whether $a = 1$ or $a = 0$. Given its uncertainty, the trader chooses his aggressiveness by balancing his expected profits conditional on two possible states. There are two lines of conflicting thought concerning the relative weight across states. On the one hand, he is aware that he would lose relatively less (more) by choosing his aggressiveness away from the optimal one at $a = 1$ ($a = 0$) due to lower (higher) price impact. This line of thought tilts his trading aggressiveness toward the optimal one conditional on $a = 0$. On the other hand, being Bayesian-rational, he is also aware that he is more likely to be in the market as one of many traders than to be the only one. Hence, the other line of thought goes in the way that his trading aggressiveness is tilted toward the optimal one conditional on $a = 1$. However, the former line of thought is still "dominant" in the sense that he tends to tilt his trading aggressiveness toward the optimal one conditional

[22]Proposition 1 symmetrically implies $\nu_1/L_S > \nu_0$ in the benchmark case. Accordingly, ν_1 must decrease and ν_0 must increase so that $I_1(\nu_1) > 0$ and $I_0(\nu_0) < 0$ in the main case, where $\nu_1/L_S = \nu_0$ must hold.

on $a = 0$, thereby disproportionately changing his trading aggressiveness conditional on $a = 1$ by trading more aggressively (less aggressively) when liquidity is driven by informed trading (noise trading). The unambiguous dominance of the former economic force, which drives the dominance of change in trading volume conditional on $a = 1$, is contrasted with the mixed direction of dominance in determining each *individual* trader's trading aggressiveness in Lemma 1. This can be understood by noting (i) the former economic force predominantly influencing each trader's aggressiveness conditional on $a = 1$ and (ii) each trader's aggressiveness at $a = 1$ being disproportionally weighted in (total) trading volume compared with that at $a = 0$ as seen in Equation (5).

4.2 Effect of market-depth uncertainty on expected price informativeness

Price informativeness measures the quality of information contained in the equilibrium price. Drawing on the idea that firm investment is an option on information and firm value embeds the value of this option, we follow Bai et al. (2016) to define price informativeness formally. It can be regarded as a part of aggregate welfare, reflecting shareholders' benefit from the information in the price. While it is alternatively defined in many previous studies (e.g., Lou and Rahi (2023), Rostek and Weretka (2012)) as the (normalized) conditional variance of the asset value, their definition is ordinally equivalent to ours.[23] Moreover, this welfare-based approach enables us to further interpret its expectation over states as part of expected aggregate welfare, which is a standard welfare criterion under uncertainty.

Definition 4. *For given state $a \in \{0, 1\}$, price informativeness is defined as the variance of conditional expectation of the asset value from the viewpoint of agents observing the equilibrium price p and the state a:*

$$PI = Var\left(\mathbb{E}[\theta|p, a]|a\right),$$

[23] It follows from the law of total variance that $\text{Var}(\mathbb{E}[\theta|p, a]) = \text{Var}(\theta) - \mathbb{E}(\text{Var}[\theta|p, a]) = \text{Var}(\theta) - \text{Var}[\theta|p, a]$, where the second equality holds because the conditional variance is constant due to the normality of random variables. This implies that our definition based on the variance of conditional expectation is one-to-one mapped with their definition based on the conditional variance of the asset value.

where the variance is taken with respect to the asset value θ and the noise order ω. In addition, expected price informativeness $\mathbb{E}[PI]$ is the prior-weighted average of price informativeness over states $a \in \{0,1\}$. Denote by $\mathbb{E}[PI_b]$ and $\mathbb{E}[PI_m]$ the expected price informativeness in the benchmark case and the main case, respectively.

The above definition of expected price informativeness slightly extends the corresponding definition in Bai et al. (2016) to the situation where price informativeness may vary across $a \in \{0,1\}$ in the current model. In particular, we first define PI conditional on each state a, which coincides with Bai et al. (2016)'s definition of price informativeness with outside agents looking at the price knowing the realized state a. Then we take its state-wise average $\mathbb{E}[PI]$ to represent their expected aggregate welfare, which can be viewed as a measure of real efficiency from the informational role of financial prices.

Given our welfare-based definition of price informativeness, it is bounded by a finite value at which the price fully aggregates all private information held by strategic traders without any error. This concept of full information aggregation coincides with the definition of "privately revealing" equilibrium price termed in Rostek and Weretka (2012).

To get price informativeness in each state $a \in \{0,1\}$, we first note that the equilibrium price is informationally equivalent to the aggregate demand y. Thus, we have

$$PI = \mathrm{Var}(\mathbb{E}[\theta|p,a]|a) = \mathrm{Var}(\mathbb{E}[\theta|y,a]|a) = \mathrm{Var}\left(\theta_0 + \lambda^* \left(\sum_{i=1}^{L_S} X_i^*(a_i,\mathcal{I}_i) + M_N\omega\right)\right),$$

where λ^* is the value of λ given by Equation (2) in equilibrium (i.e., at $\beta = \beta^*$). Using Equation (2) and $X_i^*(a_i,\mathcal{I}_i) = \beta^*(\theta - \theta_0)$, price informativeness conditional on each state a is given by

$$PI = \frac{(\sigma_0^2)^2}{\sigma_0^2 + \frac{M_N^2\sigma_\omega^2}{M_S^2(\beta^*)^2}}. \tag{9}$$

It inversely reflects the ratio of volume between noise trading (i.e., $M_N^2\sigma_\omega^2$) and informed trading (i.e., $M_S^2(\beta^*)^2$), which represents the relative size of noise of aggregate demand from the viewpoint of the market-maker.

In the benchmark case, we apply Proposition 1 to Equation (9) for $\beta^* = \beta_b^*$ to obtain

$$PI_b = \frac{M_S}{M_S+1}\sigma_0^2. \tag{10}$$

It is easy to see that PI_b is increasing in the number of participating strategic traders M_S toward its level at which the price fully aggregates all private information held by strategic traders, and it does not change as the number of participating noise traders M_N increases. Intuitively, the former follows from the observation that the proportion of informed trading relative to noise trading is increasing in the number of participating strategic traders and eventually becomes dominant, as seen in Equation (5) in the previous subsection. Also, the latter arises from the fact that strategic traders trade more aggressively in response to an increase in the number of participating noise traders due to enhanced liquidity, thereby fully offsetting the decrease of PI_b directly resulting from increased noise trading. Accordingly, expected price informativeness is its prior-weighted average over states $a \in \{0, 1\}$:

$$\mathbb{E}[PI_b] = \left(\frac{1}{2} \cdot (1 - q) + \frac{L_S}{L_S + 1} \cdot q \right) \sigma_0^2. \tag{11}$$

As we move from the benchmark case to the main case, note first that the (relative) volume of informed trading changes in Equation (5) because each strategic trader's aggressiveness changes from β_b^* to β_m^* for each state $a \in \{0, 1\}$. In parallel with the corresponding change in expected trading volume analyzed in the previous subsection, Proposition 2 and Equation (9) for $\beta^* = \beta_m^*$ imply that price informativeness changes conditional on each state a as follows:

1. When liquidity is driven by informed trading (i.e., $L_S > L_N^2$), higher (lower) optimal aggressiveness of strategic traders leads to higher (lower) price informativeness conditional on the state of high liquidity $a = 1$ (low liquidity $a = 0$).

2. When liquidity is driven by noise trading (i.e., $L_S < L_N^2$), lower (higher) optimal aggressiveness of strategic traders leads to lower (higher) price informativeness conditional on the state of high liquidity $a = 1$ (low liquidity $a = 0$).

Similarly to expected trading volume analyzed in the previous subsection, the comparison of expected price informativeness between the benchmark case and main case hinges on the comparison between the change of price informativeness conditional on $a = 0$ and that conditional on $a = 1$. In contrast to the comparison of expected trading volume in Proposition 3, the following proposition shows that its change conditional on $a = 0$ is unambiguously dominant in determining the sign of change in expected price informativeness.

Proposition 4. *For every $L_S \geq 2$ and $L_N \geq 1$, the following statements hold:*

(i) When liquidity is driven by informed trading (i.e., $L_S > L_N^2$), expected price informativeness in the main case is lower than that in the benchmark case.

(ii) When liquidity is driven by noise trading (i.e., $L_S < L_N^2$), expected price informativeness in the main case is higher than that in the benchmark case.

The sign of change in expected price informativeness (i.e., $\mathbb{E}[PI_m] - \mathbb{E}[PI_b]$) is symmetric to that in expected trading volume in Proposition 3, depending on whether liquidity is driven by informed trading ($L_S > L_N^2$) or noise trading ($L_S < L_N^2$).

Note first that price informativeness conditional on each state a is determined so as to ensure that the resulting price is close to the asset value to the extent that each trader has no incentive to be further aggressive. Formally, conjecturing that every trader's aggressiveness is β, we can rewrite each trader's first-order condition in Equation (8) as follows:

$$\left(1 - \mathbb{E}\left[\frac{\zeta_a}{\sigma_0^2}\frac{M_S+1}{M_S}\middle|\mathcal{I}_i\right]\right)(\theta - \theta_0) = 0,$$

where ζ_a denotes price informativeness conditional on state $a \in \{0,1\}$ given by Equation (9). Similarly to the analysis of expected trading volume in the previous subsection, we can think of the left-hand side as a function of ζ_a for each state $a \in \{0,1\}$. Denote by $J_a = J_a(\zeta_a)$ the left-hand side conditional on each a normalized over $\theta - \theta_0$, i.e.,

$$J_a(\zeta_a) := 1 - \frac{\zeta_a}{\sigma_0^2}\frac{M_S+1}{M_S},$$

where $M_S = 1 + a(L_S - 1)$. It is intuitive that $J_a(\zeta_a)$ is decreasing in ζ_a for each a because higher price informativeness means less trading opportunity for each trader. In the benchmark case, the expectation in the above equation is equal to the true value, implying that ζ_a is determined for each a, i.e., $J_a(\zeta_a) = 0$. In the main case, the expectation is equal to the participation-conditional-probability-weighted average over states. Thus, each trader's first-order condition in the main case can be written as

$$\Pr(a = 1|\mathcal{I}_i)J_1(\zeta_1) + \Pr(a = 0|\mathcal{I}_i)J_0(\zeta_0) = 0,$$

pinning down β_m^* and thereby ζ_0 and ζ_1 by Equation (9).

When liquidity is driven by informed trading (i.e., $L_S > L_N^2$), ζ_0 decreases (i.e., $PI_b > PI_m$ at $a = 0$) and ζ_1 increases (i.e., $PI_b < PI_m$ at $a = 1$) as we move from the benchmark case to the main case. It then follows that $J_0(\zeta_0) > 0$ and $J_1(\zeta_1) < 0$ in the main case, where their participation-conditional-probability-weighted average must be zero. The proposition indicates that $\mathbb{E}[\zeta_a]$ is lower in the main case due to the dominance of the decrease in ζ_0, which represents price informativeness conditional on $a = 0$. In parallel with the analysis of change in expected trading volume in Subsection 4.1, there are two opposing economic forces at work: (i) J_1 is decreasing in ζ_1 with a flatter slope than J_0 does so in ζ_0 due to lower price impact at $a = 1$ compared with $a = 0$. This effect moves $\mathbb{E}[\zeta_a]$ in the positive direction in response to a change from the benchmark case to the main case. (ii) As the probability weights are actually tilted toward $a = 1$ compared with prior ones (i.e., $\Pr(a = 1|\mathcal{I}_i) > q$), the decrease in ζ_0 becomes larger and the increase in ζ_1 becomes smaller. This effect moves $\mathbb{E}[\zeta_a]$ in the negative direction in response to a change from the benchmark case to the main case. According to the proposition, the second force (ii) is dominant, leading to the dominance of decrease in price informativeness at $a = 0$.

When liquidity is driven by noise trading (i.e., $L_S < L_N^2$), ζ_0 increases (i.e., $PI_b < PI_m$ at $a = 0$) and ζ_1 decreases (i.e., $PI_b > PI_m$ at $a = 1$) as we move from the benchmark case to the main case. It then follows that $J_0(\zeta_0) < 0$ and $J_1(\zeta_1) > 0$ in the main case, where their participation-conditional-probability-weighted average must be zero. As in the above case of liquidity driven by informed trading, the proposition tells us that $\mathbb{E}[\zeta_a]$ increases due to the dominance of the increase in ζ_0, which stems from the dominance of the economic force through participation-conditional probabilities of states.

The changes in expected trading volume and price informativeness have opposite signs even though they are subject to the same two economic forces as described above. Their opposite signs arise from the fact that the (first) economic force from asymmetric price impacts across states influences expected price informativeness weakly compared with its influence on expected trading volume. Formally, compared with the corresponding asymmetry of slopes across I_0 and I_1 behind the change in expected trading volume in Subsection 4.1, the asymmetry of slopes across J_0 and J_1 is less pronounced, relatively weakening the influence of (first) economic force on expected price informativeness via tilting each trader's aggressiveness toward $a = 0$. The less flatter functional form of $J_a = J_a(\zeta_a)$ at $a = 1$ arises from the boundedness of ζ_a, which corresponds to that of (expected) price informativeness inherent in its welfare-based definition.

5 Market size and its state-wise variation

In this section, we explore the pattern of expected trading volume and price informativeness under state-wise variation of market size. In Subsection 5.1, we first analyze the effects of the state-wise variation of market size. In Subsection 5.2, we investigate how these effects confound the effects of (average) market size, thereby influencing qualitative properties of expected trading volume and price informativeness.

5.1 Effects of state-wise variation of aggregate participation

In order to parametrize the state-wise variation of aggregate participation, we fix constants $C_S \geq 2$ and $C_N \geq 1$ representing the average numbers of strategic and noise traders, respectively, and set L_S and L_N to be functions of q denoted by $\tilde{L}_S = \tilde{L}_S(q)$ and $\tilde{L}_N = \tilde{L}_N(q)$, respectively, as follows:

$$\tilde{L}_S(q) = 1 + \frac{C_S - 1}{q} \text{ and } \tilde{L}_N(q) = 1 + \frac{C_N - 1}{q}.$$

The special case $q = 1$ corresponds to a market with fixed size having C_S strategic traders and C_N noise traders. As q decreases to $\frac{1}{2}, \frac{1}{3}, \cdots$, the model involves an uncertainty over the numbers of strategic and noise traders. Also, as q decreases further, this uncertainty becomes larger in the sense that the numbers of strategic and noise traders have higher variances, i.e.,

$$\text{Var}[M_S] = (1/q - 1)(C_S - 1)^2 \text{ and } \text{Var}[M_N] = (1/q - 1)(C_N - 1)^2.$$

Still, the average numbers of strategic and noise traders are invariant to the change in q, i.e.,

$$\mathbb{E}[M_S] = (1 - q) \cdot 1 + q \cdot \left(1 + \frac{C_S - 1}{q}\right) = C_S;$$
$$\mathbb{E}[M_N] = (1 - q) \cdot 1 + q \cdot \left(1 + \frac{C_N - 1}{q}\right) = C_N.$$

It is straightforward to extend this exercise to any real value of $q \in (0, 1]$ as long as $L_S = 1 + (C_S - 1)/q$ is an integer.[24]

At $q = 1$, there is no distinction between the benchmark case and the main case due to fixed participation of C_S strategic traders and C_N noise traders. As q decreases from one toward zero, randomness in market size appears in that the numbers of strategic and noise traders depend on the realized state $a \in \{0, 1\}$, keeping their expected numbers fixed. The changes in expected trading volume and price informativeness in the main case are given by

$$TV_m|_{q<1} - TV|_{q=1} = \underbrace{TV_m|_{q<1} - TV_b|_{q<1}}_{\text{uncertainty effect}} + \underbrace{TV_b|_{q<1} - TV|_{q=1}}_{\text{randomness effect}};$$

$$\mathbb{E}[PI_m]|_{q<1} - \mathbb{E}[PI]|_{q=1} = \underbrace{\mathbb{E}[PI_m]|_{q<1} - \mathbb{E}[PI_b]|_{q<1}}_{\text{uncertainty effect}} + \underbrace{\mathbb{E}[PI_b]|_{q<1} - \mathbb{E}[PI]|_{q=1}}_{\text{randomness effect}}.$$

These changes are decomposed into two effects. First, randomness of market size changes TV_m and $\mathbb{E}[PI_m]$ through the non-linearity of these variables in the realized numbers of participating strategic and noise traders. Second, uncertainty over the aggregate participation facing strategic traders leads to further changes in TV_m and $\mathbb{E}[PI_m]$.

In light of the two effects, the following proposition presents the sign of the resulting change in expected trading volume in each of the benchmark case and the main case.

PROPOSITION **5.** *For every $C_S \geq 2$ and $C_N \geq 1$, the following statements hold as q decreases from one toward zero:*

(i) When $1 = C_N < C_S$, TV_b decreases, TV_m increases and $TV_m > TV_b$.

(ii) When $2 \leq C_N < C_S$, both TV_b and TV_m increase. For each q, $TV_m > TV_b$ if $2 \leq \tilde{L}_N < \sqrt{\tilde{L}_S}$, and $TV_m < TV_b$ if $\sqrt{\tilde{L}_S} < \tilde{L}_N < \tilde{L}_S$.

(iii) When $2 \leq C_N = C_S$, TV_b increases, TV_m doesn't change and $TV_m < TV_b$.

(iv) When $2 \leq C_S < C_N$, TV_b increases, TV_m decreases and $TV_m < TV_b$.

Restricting attention to the main case, the resulting change in expected trading volume is twofold. First, the sign of the randomness effect is positive (negative) if $C_N \geq 2$ ($C_N = 1$).

[24]Even if $L_N = 1 + (C_N - 1)/q$ is not an integer, it is natural to think of noise trading with variance $L_N^2 \sigma_\omega^2$ (σ_ω^2) at state $a = 1$ ($a = 0$).

It follows from the fact that trading volume is convex (concave) in the "realized" market size along the $(\tilde{L}_S, \tilde{L}_N)$-line in the (M_S, M_N)-space if $C_N \geq 2$ $(C_N = 1)$.[25] Second, as seen in Proposition 3, the sign of the uncertainty effect is positive (negative) when liquidity is driven by informed trading (noise trading) (i.e., $\tilde{L}_S > \tilde{L}_N^2$ $(\tilde{L}_S < \tilde{L}_N^2)$). It follows from the dominance of the positive (negative) effect through asymmetric price impacts across states over the opposing force through participation-conditional probabilities of states. The total change in TV_m resulting from a decrease in q is the sum of the two effects. When liquidity is driven by informed trading, the uncertainty effect is dominant (i.e., when $\tilde{L}_N = 1$) or both effects are positive (i.e., when $\tilde{L}_N \geq 2$). Accordingly, TV_m increases. When liquidity is driven by noise trading, the randomness effect is positive, whereas the uncertainty effect is negative. The proposition indicates that the randomness (uncertainty) effect is dominant so that the total change of TV_m is positive (negative) when $C_S > C_N$ $(C_S < C_N)$, which corresponds to $\tilde{L}_S > \tilde{L}_N$ $(\tilde{L}_S < \tilde{L}_N)$. At $C_S = C_N$, which corresponds to $\tilde{L}_S = \tilde{L}_N$, the negative uncertainty effect (i.e., $TV_m - TV_b < 0$ given q) balances the positive randomness effect (i.e., the increase in TV_b from a decrease in q).

Similarly, the following proposition presents the sign of change in expected price informativeness.

PROPOSITION **6.** *For every $C_S \geq 2$ and $C_N \geq 1$, the following statements hold as q decreases from one toward zero:*

(i) When $1 = C_N < C_S$, both $\mathbb{E}[PI_b]$ and $\mathbb{E}[PI_m]$ decrease, and $\mathbb{E}[PI_m] < \mathbb{E}[PI_b]$.

(ii) When $2 \leq C_N < C_S$, both $\mathbb{E}[PI_b]$ and $\mathbb{E}[PI_m]$ decrease. For each q, $\mathbb{E}[PI_m] < \mathbb{E}[PI_b]$ if $2 \leq \tilde{L}_N < \sqrt{\tilde{L}_S}$, and $\mathbb{E}[PI_m] > \mathbb{E}[PI_b]$ if $\sqrt{\tilde{L}_S} < \tilde{L}_N < \tilde{L}_S$.

(iii) When $2 \leq C_N = C_S$, $\mathbb{E}[PI_b]$ decreases, $\mathbb{E}[PI_m]$ doesn't change and $\mathbb{E}[PI_m] > \mathbb{E}[PI_b]$.

[25] If the number of noise traders is fixed (i.e., $C_N = 1$), trading volume is concave in the realized and publicly known number of participating strategic traders because each individual strategic trader chooses to be less aggressive with a higher realized number of them, as implied by Proposition 1. By contrast, if $C_N \geq 2$, a higher realized number of participating strategic traders comes with a higher realized number of participating noise traders so that the latter may incentivize each individual strategic trader to trade more aggressively, overturning the aforementioned concavity of trading volume. Indeed, this is the case for every $C_N \geq 2$.

(iv) When $2 \leq C_S < C_N$, $\mathbb{E}[PI_b]$ *decreases,* $\mathbb{E}[PI_m]$ *increases and* $\mathbb{E}[PI_m] > \mathbb{E}[PI_b]$.

First, the sign of the randomness effect is always negative due to the concavity of price informativeness in the realized number of strategic traders M_S in Equation (10). Second, as seen in Proposition 4, the sign of the uncertainty effect is negative (positive) whenever liquidity is driven by informed trading (noise trading) (i.e., $\tilde{L}_S > \tilde{L}_N^2$ ($\tilde{L}_S < \tilde{L}_N^2$)). It follows from the dominance of the negative (positive) effect through asymmetric price impacts across states over the opposing force through participation-conditional probabilities of states. The total change of $\mathbb{E}[PI_m]$ resulting from a decrease in q is the sum of the two effects. When liquidity is driven by informed trading, both effects are negative, meaning that $\mathbb{E}[PI_m]$ decreases. When liquidity is driven by noise trading, the uncertainty effect turns to be positive, while the randomness effect is still negative. The proposition indicates that the randomness (uncertainty) effect is dominant so that the total change in $\mathbb{E}[PI_m]$ is negative (positive) when $C_S > C_N$ ($C_S < C_N$), which corresponds to $\tilde{L}_S > \tilde{L}_N$ ($\tilde{L}_S < \tilde{L}_N$). At $C_S = C_N$, which corresponds to $\tilde{L}_S = \tilde{L}_N$, the positive uncertainty effect (i.e., $\mathbb{E}[PI_m] - \mathbb{E}[PI_b] > 0$ given q) balances the negative randomness effect (i.e., the decrease in $\mathbb{E}[PI_b]$ from a decrease in q).

Overall, the effects of state-wise variations of market size on expected trading volume TV_m and expected price informativeness $\mathbb{E}[PI_m]$ generally hinge on whether liquidity is driven by informed or noise trading, highlighting the role of the uncertainty effect analyzed in Subsections 4.1 and 4.2. The only further complication arises from the additional randomness effect. However, this additional effect changes only the specific condition for the relative ratio between informed trading and noise trading to pin down the signs of changes in TV_m and $\mathbb{E}[PI_m]$, rather than altering their qualitative pattern.

The following simulations (Figure 1) are consistent with the results depending on whether liquidity is driven by informed trading with $C_N = 1$ (two graphs on the first row) or $C_N \geq 2$ (two graphs on the second row) or liquidity is driven by noise trading (two graphs on the third row). In these simulations, we take $\sigma_0^2 = \sigma_\omega^2 = 1$ and θ_0 is irrelevant.

5.2 Effects of changing market size

Without any state-wise variation of aggregate participation (i.e., $q = 1$), our framework with symmetric and risk-neutral strategic traders would provide fairly standard implications con-

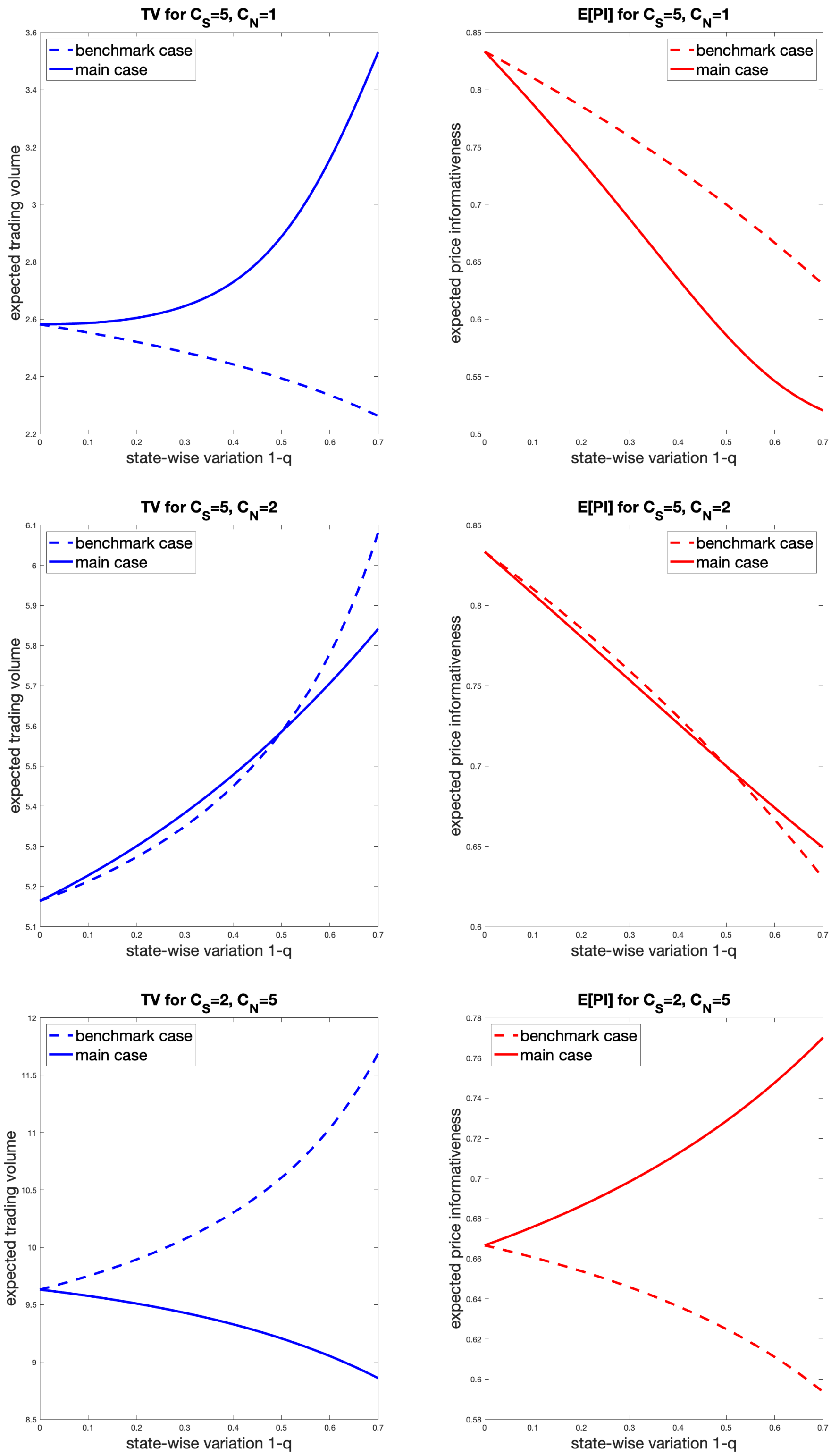

Figure 1: Effects of state-wise variation of aggregate participation (i.e., $1-q$): $C_S = 5$ and $C_N = 1$ (first row), $C_S = 5$ and $C_N = 2$ (second row), $C_S = 2$ and $C_N = 5$ (third row)

cerning the effect of changes in the numbers of strategic and noise traders. That is, Equations (6) and (10) imply that increasing the number of strategic traders (i.e., L_S) increases both expected trading volume and price informativeness, whereas increasing the number of noise traders (i.e., L_N) increases expected trading volume and does not change expected price informativeness. The increases in expected trading volume naturally arise from the general idea that increasing market size enhances liquidity, thereby unambiguously increasing the aggregate incentive to trade an asset despite a possible reduction in *individual* trading aggressiveness.[26] The overall increase in expected price informativeness also follows from the same idea.

Even with a state-wise variation of market size (i.e., $0 < q < 1$), the same arguments continue to hold in the benchmark case, which is equivalent to fixed market size with M_S strategic traders and M_N noise traders for each realized state $a \in \{0, 1\}$ in terms of each strategic trader's trading aggressiveness.

However, our analysis with uncertain market depth (i.e., the main case) in Subsection 5.1 suggests that the above standard properties do not necessarily hold.

Proposition 7. *As we change each of L_S, L_N, and q having other variables fixed, the following statements hold.*

(i) For large L_S, increasing L_S increases TV_m. However, in this limit, it decreases $\mathbb{E}[PI_m]$ if $0 < q < \frac{1}{2}\left(1 + \sqrt{\frac{L_N^2}{L_N^2+4}}\right)$, and increases $\mathbb{E}[PI_m]$ if $\frac{1}{2}\left(1 + \sqrt{\frac{L_N^2}{L_N^2+4}}\right) < q < 1$.

(ii) Increasing L_N always increases TV_m and $\mathbb{E}[PI_m]$.

(iii) Increasing q increases $\mathbb{E}[PI_m]$. However, for large L_S, as q increases from zero toward one, TV_m increases for $q \in \left(0, \frac{1}{3}\right)$, decreases for $q \in \left(\frac{1}{3}, \frac{3}{4}\right)$, and then increases for $q \in \left(\frac{3}{4}, 1\right)$. On the other hand, for large L_N, as q increases from zero toward one, TV_m always increases.

Most notably, as the number of strategic traders L_S increases, expected price informativeness may decrease in contrast to the corresponding change in price informativeness with fixed participation discussed above. The decrease in expected price informativeness is attributed to

[26] The "aggregate" incentive to trade the asset refers to each trader's incentive to trade given the aggregate demand as seen in Equation (4). See Park (2024) for a formal description.

an effective increase in the state-wise variation of overall participation of traders. Accordingly, the additional uncertainty over market depth causes a confounding effect on expected price informativeness together with the direct effect of (average) market size.

Formally, the main case is denoted by $L = (q, L_S, L_N)$ following the notation of our model in Section 2. Then its corresponding model with fixed participation, which is a special case of our model with $q = 1$, is similarly denoted by $C = (1, 1 + q(L_S - 1), 1 + q(L_N - 1))$.[27] Consider an increase of L_S by Δ. The resulting change in expected price informativeness is given by

$$\begin{aligned}\mathbb{E}[PI_m]|_{L'} - \mathbb{E}[PI_m]|_L = & \underbrace{PI|_{C'} - PI|_C}_{\text{average market size}\uparrow} \\ & + \underbrace{(\mathbb{E}[PI_m]|_{L'} - PI|_{C'})}_{\text{post-change variation}} - \underbrace{(\mathbb{E}[PI_m]|_L - PI|_C)}_{\text{pre-change variation}},\end{aligned}$$

where $L' = (q, L_S + \Delta, L_N)$ and $C' = (1, 1 + q(L_S - 1) + q\Delta, 1 + q(L_N - 1))$. The first line represents the change of (expected) price informativeness attributed to the increase in average market size, which is unambiguously positive. On the second line, the first term represents the part of post-change expected price informativeness attributed to the state-wise variation of aggregate participation in line with our analysis in Subsection 5.1. Likewise, the second term represents the corresponding part of pre-change expected price informativeness. Hence, their difference on the second line contains the change of expected price informativeness attributed to an effective increase in the state-wise variation of aggregate participation stemming from the increase of L_S by Δ.[28] As discussed above, the first line is always positive. However, as L_S is sufficiently large, the confounding effect on the second line can be negative in line with Proposition 6 (i.e., Parts (i) and (ii)) indicating a negative effect of state-wise variation of aggregate participation on expected price informativeness. As seen in the proposition, the possibility of dominance of the negative confounding effect occurs for large L_S when the high-liquidity probability q is not too close to one.

Such decrease in expected price informativeness seems to be less likely to be the case when L_S is relatively small, say $L_S = 2$. In this case, as seen in Equation (10) for fixed participation

[27] While the formal definition is impeded by the integer constraint, it could be understood by restricting attention to the case where $q(L_S - 1)$ is an integer in parallel with our analysis in Subsection 5.1.

[28] Its interpretation is further complicated by the non-separability of the change of average market size (from C to C') and the change of state-wise variation in terms of their effects on expected price informativeness.

(i.e., $q = 1$ so that $M_S = L_S$), the positive direct effect through average market size is relatively large compared with the corresponding effect in the limit of large L_S, making it less likely that expected price informativeness decreases due to the negative confounding effect described above. While it is analytically challenging to determine the sign of changes in expected trading volume and price informativeness, we have found by simulations that both expected trading volume and price informativeness increase when L_S increases from 2 to 3 for various reasonable parameters of the model.[29]

Combining these together, we can naturally expect that expected price informativeness is hump-shaped in L_S as long as the high-liquidity probability q is not too large (i.e., $0 < q < \frac{1}{2}\left(1 + \sqrt{\frac{L_N^2}{L_N^2+4}}\right)$). This property follows from the conjecture that the negative confounding effect balances the positive direct effect at an intermediate threshold level of L_S, and, above that level of L_S, the former becomes dominant, as formally verified in Proposition 7 only in the limit of large L_S. Indeed, in the upper-right graph in Figure 2, our simulation result confirms the property for a set of parameters of the model, illustrating the optimal market size in terms of L_S for each $q \in \{0.1, 0.3, 0.5, 0.7\}$. For the rest of parameters, we use $\sigma_0^2 = \sigma_\omega^2 = 1$ and $L_N = 5$. According to the simulation result, the price-informativeness-maximizing level of L_S crucially depends on the high-liquidity probability q, which might differ across specific assets in practice.

As seen in the upper-left graph in Figure 2, the highlighted decrease in expected price informativeness comes with higher trading volume. As L_S increases, the change in expected trading volume can similarly be decomposed into its change attributed to increased average market size and that attributed to the additional state-wise variation of aggregate participation. As discussed above, the former is always positive. Moreover, when L_S is large, the latter confounding effect is positive as well in line with Proposition 5 (i.e., Parts (i) and (ii)) indicating a positive effect of state-wise variation of aggregate participation on expected trading volume. In this limit, Lemma 1 and Equation (5) imply

$$TV_m \to \begin{cases} L_S q\sqrt{1-2q}\sqrt{\frac{2}{\pi}\sigma_\omega^2} & \text{if } q < \frac{1}{2}; \\ \sqrt{L_S}L_N\sqrt{\frac{q^3}{2q-1}}\sqrt{\frac{2}{\pi}\sigma_\omega^2} & \text{if } q > \frac{1}{2}. \end{cases} \tag{12}$$

[29] We have not found any numerical example where either expected trading volume or price informativeness decreases as L_S increases from 2 to 3.

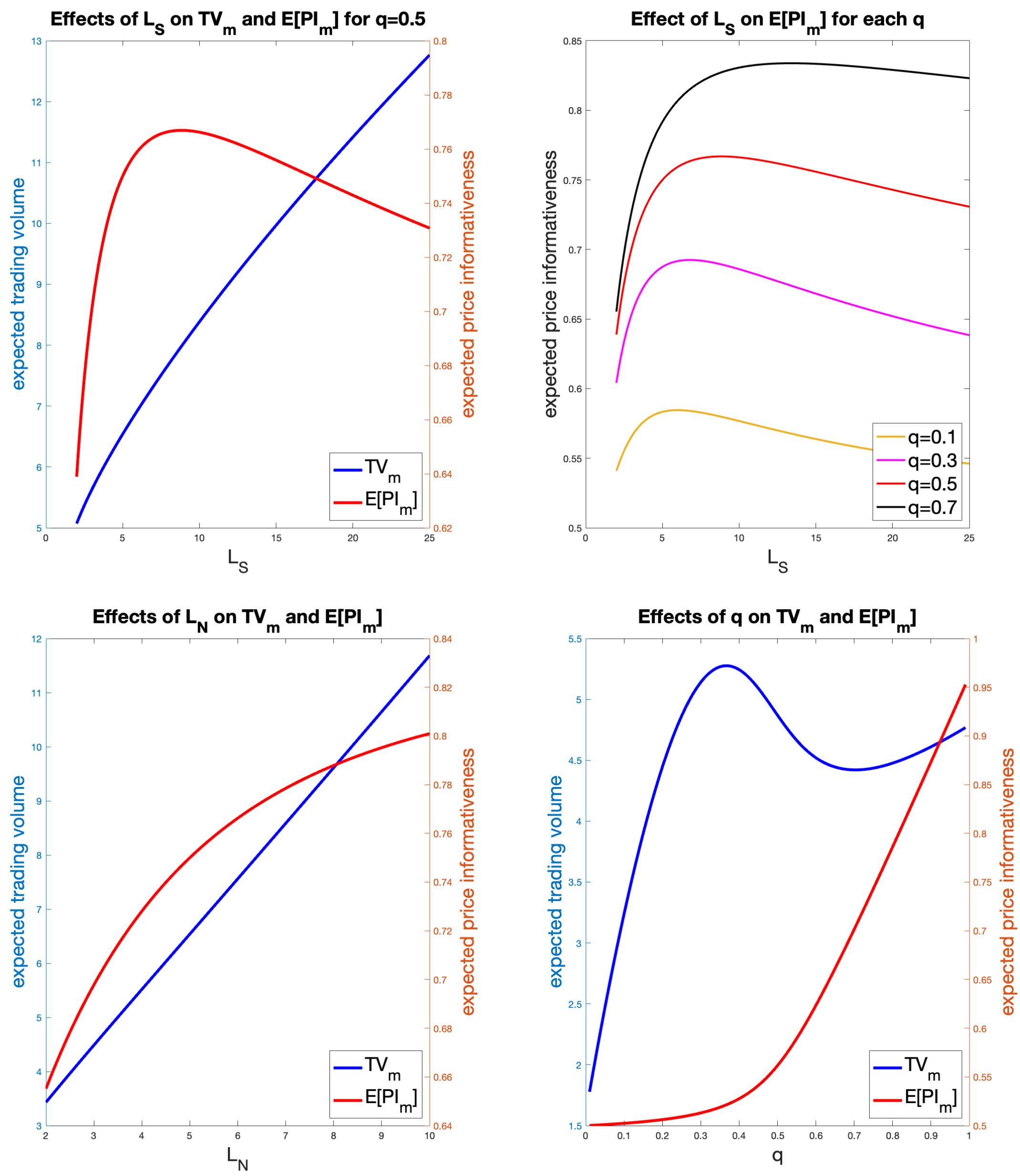

Figure 2: Numerical results of TV_m and $\mathbb{E}[PI_m]$ across market size: Change in L_S (upleft, $L_N = 5$ and $q = 0.5$) and that for $q = 0.1, 0.3, 0.5, 0.7$ (upright, $L_N = 5$), change in L_N (downleft, $L_S = 5$ and $q = 0.5$), change in q (downright, $L_S = 25$ and $L_N = 1$)

For $q < \frac{1}{2}$, it is of higher order (i.e., $\mathcal{O}(L_S)$) than those in the benchmark case and with fixed participation (i.e., $q = 1$), which corresponds to $\mathcal{O}(\sqrt{L_S})$ as implied by Equation (6). Even for $q > \frac{1}{2}$, it is still higher despite the same order (i.e., $\mathcal{O}(\sqrt{L_S})$) as implied by Proposition 3.

As the number of noise traders L_N increases, expected trading volume and price informativeness always increase. With fixed market size (i.e., $q = 1$), Equation (6) implies that the former would increase but Equation (10) implies that the latter would not change because strategic traders adjust their aggressiveness proportionally in response to noise trade. With market-depth uncertainty (i.e., $q < 1$), the increase in L_N also leads to an increase in the state-wise variation of market size, thereby strictly increasing expected price informativeness in line with statement (iv) in Proposition 6. On the other hand, while the same confounding effect via the state variation of market size is negative on expected trading volume in line with statement (iv) in Proposition 5, it does not overturn the sign of overall change in expected trading volume. These unambiguous increases in expected trading volume and price informativeness are confirmed in our simulation result in the down-left graph in Figure 2.

As the high-liquidity probability q increases, expected price informativeness always increases. However, this increase is not necessarily attributed to more informed trading by strategic traders, which would lead to higher trading volume at the same time. Indeed, the above expression of expected trading volume for large L_S implies that it decreases for $q \in \left(\frac{1}{3}, \frac{3}{4}\right)$. Here, an increase in q involves not only a direct increase in the average market size but also a change in the state-wise variation of aggregate participation, which is most pronounced in the middle of range of $q \in (0, 1)$. While the former direct effect through average market size unambiguously increases both expected trading volume and price informativeness, the latter effect is rather mixed throughout the range of $q \in (0, 1)$. Most notably, expected trading volume may decrease by q in the upper-middle range of $q \in (0, 1)$, where a marginal increase in q *reduces* the state-wise variation of aggregate participation and thereby decreases expected trading volume for large L_S in line with Proposition 5 (i.e., Parts (i) and (ii)) indicating a negative effect of *decreasing* the state-wise variation of aggregate participation on expected trading volume. It could be explained with the (reverse) confounding effect via an effective decrease in the state-wise variation of aggregate participation. For large L_S, where the direct positive effect through average market size is small as noted above, such confounding effect is dominant. In contrast to the possibility of non-monotonicity of expected trading volume, the proposition indicates that the sign of overall change in expected price informativeness cannot be overturned by the confounding effect via the state-wise variation of aggregate participation for any $q \in (0, 1)$. These properties are confirmed in our simulation result in the down-right graph in Figure 2.

With the assumption of common per-unit value and common prior across strategic informed traders, it seems rather standard in the literature that trading volume and price informativeness tend to be increasing in market size. For example, in the current framework based on Kyle (1985) and its extension to complex information structures, such positive relation between market size and price informativeness is confirmed by Lambert et al. (2018) in the limit of large market size.[30] According to the literature beyond the current framework based on Kyle (1985), the possibility of complex relation among these variables has been explained with heterogeneous valuations across strategic traders (e.g., Lee and Kyle (2022), Lou and Rahi (2023), Rostek and Weretka (2012)). In the same spirit, various possible outcomes concerning price informativeness can be obtained with belief disagreements (e.g., Davila and Parlatore (2021), Park (2024)) or other forms of heterogeneity in trader characteristics (e.g., Kacperczyk et al. (2024)). Without such heterogeneity across traders, the literature appears to suggest going beyond a single-market framework, such as incorporating cross-exchange interaction (e.g., Chen and Duffie (2021)), to deviate from the standard argument that market expansion (fragmentation) increases (decreases) trading volume and price informativeness.

In contrast, our results are based on a single-market framework with common per-unit value and common prior and information across strategic traders, thereby providing a clear case where market expansion (fragmentation) may weaken (strengthen) the role of prices as information aggregators for outside decision-makers. In our model, as L_S increases, an increase in average liquidity drives the positive direct effects of average market size on expected trading volume and price informativeness.[31] Nevertheless, the possibility of negative relation between market size and expected price informativeness and the overall discrepancy between expected trading

[30] Their only caveat is the mild assumption that noise trade is uncorrelated or positively correlated with the asset value.

[31] Ruling out uncertainty over market depth, Proposition 1 implies that liquidity (i.e., the inverse of Kyle's lambda) is increasing in (unchanged with) L_S at state $a = 1$ ($a = 0$).

volume and price informativeness are attributed to the state-wise variation of market size as seen in Propositions 5 and 6 for large L_S.

6 Further implications

6.1 Relation between trading volume and price volatility

The empirical relation between trading volume and price volatility has received considerable interest in the literature. On the one hand, early studies document a contemporaneous on-average positive relation between trading volume and price volatility (e.g., Karpoff (1987), Jones et al. (1994)). On the other hand, more recent studies document the presence of volumes which cannot be explained with price volatility.[32] That is, large trading volume can be observed even without price movements.

In standard equilibrium frameworks of financial markets (e.g., Kyle (1985)), the former type of volumes that are associated with volatility can be explained with informed agents' trades, which reflect their private information and thus result in price movements. The latter type of volumes that do not cause price movements can be modeled as exogenous noise trading, which may arise from life cycle and risk-hedging.[33] However, it has been argued that these noise-creating trading motives are insufficient to explain large trading volume in practice (e.g., Hong and Stein (2007)). Accordingly, there is a long-held consensus that belief disagreements across traders are required to explain large trading volume without substantial price movements. Indeed, the model of traders with heterogeneous priors in Kandel and Pearson (1995) and its reduced-form framework in Bollerslev et al. (2018) indicate that the positive volume-volatility relation tends to be weaker with larger belief disagreements.

Extending Kyle (1985), our model with uncertain market depth can explain a weak or even negative volume-volatility relation together with abnormally large trading volume even without

[32] Empirical evidence is particularly well-documented in the context of trading volume and price movements following the release of new information, such as earnings and FOMC announcements (e.g., Kandel and Pearson (1995), Bollerslev et al. (2018)).

[33] See Foster and Viswanathan (1990) and He and Wang (1995) for such frameworks based on noise trading to explain the weak relationship between volume and volatility.

belief disagreements. To begin with, we formally define (state-conditional) price volatility and its expectation as follows:

Definition 5. *Price volatility is the state-conditional variance of equilibrium price* $VOL := Var(p|a)$. *Also, expected price volatility is its state-wise expectation* $\mathbb{E}[VOL]$.

Our definition of price volatility as the idiosyncratic variance is consistent with previous studies (e.g., Davila and Parlatore (2023)). While volatility is inherently a dynamic concept, the current definition could be non-substantially extended to a "repeated static economy" with short-lived private information as in Davila and Parlatore (2023). On the other hand, expected price volatility reflects the cross-sectional and/or time-series average of price volatility, which varies across assets and time in practice due to the randomness of realized state.

Price volatility defined above is equivalent to price informativeness in the current model. Formally, we have $VOL = \text{Var}\left(\mathbb{E}[\theta|a,y]|a\right) = PI$ for each state $a \in \{0,1\}$. This relation is commonly adopted in the empirical literature.[34] For example, Campbell et al. (2023) stated that "idiosyncratic volatility serves as an empirical proxy for the flow of firm-specific information." Likewise, its expectation is equal to expected price informativeness. Given the equivalence between price volatility and informativeness, it naturally follows that the relation between trading volume and (expected) price volatility is positive across states $a \in \{0,1\}$ in the benchmark case.[35]

Now suppose that a sample of observations features a variation of uncertainty over the aggregate participation. As seen throughout Sections 4 and 5, such variation could arise from changes in the availability of state-relevant information (Section 4), changes in the extent of variation of aggregate participation (Subsection 5.1), and market expansion or fragmentation (Subsection 5.2). Combined with the equivalence between price informativeness and volatility, our results imply that subsamples of observations with different levels of uncertainty over the aggregate participation will present a negative cross-subsample correlation between trading volume and price volatility. For example, if some event causes an uncertainty over aggregate

[34] On the theoretical side, it is not entirely general beyond the current framework (e.g., Davila and Parlatore (2023)).

[35] Formally, as we move from $a = 0$ to $a = 1$, M_S and M_N increase strictly and weakly, respectively. Then Equation (6) implies that TV_b increases, and Equation (10) implies that $\mathbb{E}[VOL] = \mathbb{E}[PI_b]$ increases as well.

participation and liquidity is driven by informed trading (noise trading), post-event observations tend to have higher (lower) trading volume and lower (higher) expected price volatility compared with pre-event observations (Propositions 3 and 4). Therefore, these post-event observations can be empirically captured as abnormally high (low) trading volume not explained by expected price volatility.

Moreover, our results imply that the former case of relatively high post-event trading volume is likely to be dominant, driving an increase in trading volume on average following the event. Specifically, in the former case, the increased trading volume may be increasing in market size qualitatively at a steeper pace compared with the corresponding pre-event volume, whereas, in the latter case, the decreased trading volume is still increasing in market size qualitatively at the same pace as before.[36] Hence, even a relatively small fraction of such observations corresponding to large L_S in our model may result in an increase in trading volume on average throughout the entire sample.

Overall, the main results of our analysis can explain empirically relevant features of large and abnormal trading volume that cannot be explained by price volatility. Admitting that our model abstracts from belief disagreements which would alternatively cause similar features of trading volume, we suggest that these results complement the existing explanation in the literature based on belief disagreements.

Last, let us discuss the possibility of empirically testing our explanations on the pattern of trading volume and prices. The key prediction from our analysis is that the effects of uncertainty over market depth on trading volume and price informativeness hinge on whether liquidity is driven by informed or noise trading. The empirical challenge is twofold. First, the measurement of market-depth uncertainty facing informed market participants depends on proxies based on realized liquidity measures, rather than being direct. Bali et al. (2014) used a deviation of realized liquidity from its average level in the past 12 months or the conditional mean of ARMA(1,1) specification to measure stock-level liquidity shock, showing that such liquidity shock is not correctly priced in the short run. Based on their finding, we could at least

[36] As discussed in Subsection 5.2, $TV_m = \mathcal{O}(L_S)$ is of higher order than $TV_b = \mathcal{O}(\sqrt{L_S})$ for large L_S, fixed L_N, and $q < \frac{1}{2}$ by Equations (6) and (12). On the other hand, for large L_N and fixed L_S, it is formally shown by the proof of Proposition 7 and Equation (6) that $TV_m = \mathcal{O}(L_N)$ and $TV_b = \mathcal{O}(L_N)$ as well.

partially interpret such deviation from the time-series average as the extent of uncertainty over market depth. Second, the distinction between liquidity driven by informed trading and that driven by noise trading also depends on indirect proxies. In the benchmark case of the model, the state of nature with higher liquidity causes lower (higher) per-trader volume (Proposition 1) in the former (latter) scenario. Also, this relation is weakened with uncertainty over market depth, as seen in the main case of the model (Proposition 2). Thus, conditional on relatively less uncertainty over market depth, the sign of observed relation between liquidity and per-trader volume can be used to distinguish between the two scenarios. That is, its negative (positive) sign is consistent with the former (latter) scenario. Still, it might not be straightforward to observe per-trader volume, which would require investor-level transaction data, in practice.

6.2 Effects of transparency on price informativeness

In this subsection, we discuss the implications of our main results on the impact of transparency in financial markets. The theoretical literature is mostly based on REE models abstracting from market depth and restricts attention to the side of transparency concerning asset-payoff-relevant information as reviewed in Subsection 1.1. However, due to the lack of alternative sources of public information, transparency is a concern of policy relevance in small markets that typically consist of only a few institutional traders and market-maker(s), such as alternative trading systems. One non-trivial aspect of transparency in such markets is information on the aggregate participation of traders, which could be indirectly revealed through the number and volume of executed orders. The absence of such information naturally causes an uncertainty over market depth.[37]

In our model, a transparency reform concerning information on the aggregate participation of traders can be regarded a change from the main case to the benchmark case. As it corresponds to the reverse of our analysis in Section 4 (i.e., Propositions 3 and 4), it increases (decreases) expected price informativeness when liquidity is driven by informed trading (noise trading). The latter outcome occurs despite an increase in expected trading volume, which is often casu-

[37] One extreme type of opaqueness is so-called "dark pools", which refer to private trading exchanges for trading securities without being known to others. However, as they are opaque even about prices, the model does not apply in a straightforward manner.

ally regarded as an indicator of market efficiency. Taken together with the fact that expected price informativeness reflects real efficiency, this suggests that the reform may have unintended consequences of lowering real efficiency via lower price informativeness on average. According to the discussion following Proposition 4, it is mainly driven by participation-conditional probabilities of states, which are tilted toward the high-liquidity state in the main case (i.e., before the reform) but go back to the prior in the benchmark case (i.e., after the reform). As a result, when liquidity is driven by noise trading, each trader's average aggressiveness tends to decrease following the reform as it is tilted toward the prior putting more weight on the low-liquidity state.

More generally, one may consider a transparency reform concerning asset-payoff-relevant information. Examples of such policies include the mandate or standardization of public information, which would make it easier for investors to access the information. Then the above argument regarding information on the aggregate participation of traders may still hold as one of various channels of changes through enhanced transparency. For example, improving the ease of acquiring common information may make it more likely that others' participation is known to each strategic trader because it enables him to second-guess others' attentiveness to the asset. In this case, a similar conclusion is obtained regarding a potentially negative effect of transparency reforms on price informativeness when liquidity is driven by noise trading (i.e., $L_S < L_N^2$ in our model).[38]

7 Concluding remarks

The main objective of this paper is to analyze the implications of uncertainty over the overall participation of traders and the resulting market depth in financial markets. We analyze a model of strategic traders submitting market orders where they are uncertain of the overall participation of strategic and noise traders, which determines the market depth and their trading opportunities. The analysis highlights the significance of knowledge on the overall

[38]For example, consider a transparency reform as an increase in the prior probability q from q_N to q_T together with a change from the main case to the benchmark case. Then, despite the positive effect of increasing q on expected price informativeness through improved liquidity on average, this is the case in the second large-market scenario for every q_N and q_T where $0 < q_N < q_T < 1$.

participation of traders and the resulting market depth, thereby confirming our motivating argument on the potential complication of the effects of market size. In addition, the analysis can provide implications on the pattern of price volatility and the impact of transparency reforms which would not be straightforward without market-depth uncertainty.

The mechanism behind these results can be explained with two economic forces arising from asymmetric price impacts across states and participation-conditional probabilities, respectively. These forces are generally robust in the following two directions of possible extensions.

First, the two forces persist with an arbitrary distribution of the numbers of participating strategic and noise traders (i.e., M_S and M_N), which need not be perfectly correlated or binary as in the current model. In parallel with our analysis in Section 4, the influence of their two-dimensional uncertainty on expected trading volume would still generally depend on (i) the variation of slopes in $I_a(\nu_a)$ across (possibly multidimensional) states a representing the realization of M_S and M_N and (ii) participation-conditional probabilities of states.[39] The same argument applies to its influence on expected price informativeness. These suggest that the overall intuition put forth in Section 4 behind changes in expected trading volume and price informativeness would be largely unaffected even though a larger set of possible outcomes would likely arise due to higher degrees of freedom in terms of the distribution of M_S and M_N.

Second, these forces are still present in the case of strategic traders holding noisy signals.[40] Compared with the current model with strategic traders perfectly knowing the asset value, there are two additional considerations: (i) less informed trading leads to less adverse selection and thus weakens the first economic force from asymmetric price impacts across states, whereas the other one from participation-conditional probabilities of states is intact; (ii) a less-than-perfect correlation of strategic traders' errors means that new traders' entry delivers additional infor-

[39] For example, consider two possible states $a \in \{0, 1\}$ such that $a = 1$ ($a = 0$) is a state of higher proportion of M_S (M_N). As in our main analysis for $L_S > L_N^2$ in Section 4, the former leads to smaller trading opportunities for each strategic trader compared with the latter. Then, we may expect $TV_m > TV_b$ when (i) the slope of $I_a(\nu_a)$ is flatter at $a = 1$ compared with that at $a = 0$ (e.g., a larger variation of M_S across states compared with that of M_N) and (ii) the economic force through asymmetric price impacts dominates the other force through participation-conditional probabilities of states.

[40] Formally, we may consider the situation where each strategic trader has a noisy signal $s_i = \theta + \epsilon_i$, where ϵ_i is a normal error possibly correlated across traders. The analysis of this extension is available upon request from the authors.

mation, making it more likely that price informativeness increases in larger markets. Formally, our extended analysis shows that Proposition 3 is weakened by and Proposition 4 is robust to such noisy signals, as is consistent with the first argument above. Also, the possibility of decrease in price informativeness in larger market size (i.e., larger L_S) presented in Proposition 7 still (no longer) exists even with large signal errors if the correlation of signal errors across strategic traders is positive (zero) in line with the second argument above.[41]

We close the paper by discussing two broader possible extensions of interest.

First, the model may be extended to the situation where strategic traders endogenously and costly acquire their information. For example, endogenous acquisition on asset-payoff-relevant information (i.e., information on θ) might matter for our results on expected price informativeness. In particular, when $L_S > L_N^2$ ($L_S < L_N^2$), as we move from the benchmark case to the main case, the resulting decrease (increase) in expected price informativeness (by Proposition 4) might come with an increase (a decrease) in trading opportunities. If so, strategic traders would intuitively choose to acquire more (less) asset-payoff-relevant information, possibly counteracting the change in expected price informativeness. However, this idea does not necessarily hold with market-depth uncertainty in our model. In particular, when $L_S > L_N^2$ ($L_S < L_N^2$), the decrease (increase) in expected price informativeness may actually come with a decrease (increase) in the expected profit for each strategic trader.[42] It suggests that endogenous acquisition of asset-payoff-relevant information may actually reinforce the change in expected price informativeness rather than counteracting it. More broadly, an extensive analysis of information choice involving asset-payoff-relevant and aggregate-participation-relevant information, the latter of which can be regarded as an alternative way of formalizing information on market conditions considered in Ganguli and Yang (2009) and Farboodi and Veldkamp (2020), is an interesting revenue for future research.

Second, given the empirical relevance of time-varying liquidity (e.g., Chordia et al. (2000),

[41] The intuition is that higher correlation of signal errors means less "additional" information delivered by a new trader entering the market, weakening the second argument above.

[42] Formally, as L_S is large and L_N is fixed, the expected profit for each strategic trader always decreases as we move from the benchmark case to the main case. Also, as L_N is large and L_S is fixed, the expected profit for each strategic trader increases (decreases) if $L_S > 1/q$ ($L_S < 1/q$). The formal analysis and the proof of these statements are available upon request from the authors.

Huberman and Halka (2001)), the model may be extended to a dynamic setting with multiple trading rounds. Such multi-period extension deviates from our current single-period model when asset-payoff-relevant private information held by informed traders is long-lived and/or state-relevant private information held by the market-maker is long-lived. The former has long been studied with the standard assumption of constant market depth (e.g., Holden and Subrahmanyam (1992)). Moreover, the latter is considered in Hong and Rady (2002) reviewed in Subsection 1.1, suggesting that agents learn about the state from past prices and trading volume when the state is stochastically persistent (i.e., it can be inferred from past realizations). Still, it appears to be the case in practice that such learning cannot be perfect as persistent and temporary components of liquidity coexist, the latter of which would correspond to an unpredictable shock to liquidity motivating our model as described in the Introduction (Bali et al. (2014)). Thus, it is of interest to consider what would occur with the former assumption (i.e., long-lived asset-payoff-relevant private information held by informed traders) but not the latter one (e.g., assuming that the state is independently distributed in each period). We speculate that the two economic forces driving our results would largely persist in this dynamic setting as they would be present in each period regardless of the past-information-adjusted prior of asset value.

Appendix: Proofs

Proof of Proposition 1

Solving (4) yields

$$x_i = \frac{1-\lambda\beta(M_S-1)}{2\lambda}(\theta-\theta_0) = \frac{1-\frac{M_S-1}{M_S}\frac{\sigma_0^2}{\sigma_0^2+\frac{M_N^2\sigma_\omega^2}{M_S^2\beta^2}}}{\frac{2}{M_S\beta}\frac{\sigma_0^2}{\sigma_0^2+\frac{M_N^2\sigma_\omega^2}{M_S^2\beta^2}}}(\theta-\theta_0),$$

where the second equality follows from (2). Matching the coefficient in the last equality with the conjectured one gives

$$\beta_b^* = \frac{1 - \frac{M_S - 1}{M_S} \frac{\sigma_0^2}{\sigma_0^2 + \frac{M_N^2 \sigma_\omega^2}{M_S^2 (\beta_b^*)^2}}}{\frac{2}{M_S \beta_b^*} \frac{\sigma_0^2}{\sigma_0^2 + \frac{M_N^2 \sigma_\omega^2}{M_S^2 (\beta_b^*)^2}}},$$

which implies

$$\frac{M_S + 1}{M_S} \frac{\sigma_0^2}{\sigma_0^2 + \frac{M_N^2 \sigma_\omega^2}{M_S^2 (\beta_b^*)^2}} = 1. \tag{13}$$

Part (ii) follows directly from (13). The first result in Part (i) is derived from (2) and the expression for β_b^* given in Part (ii). Since $\frac{\sqrt{M_S}}{M_S+1} \frac{1}{M_N}$ is strictly decreasing in M_S and M_N, the second result in Part (i) follows. The proof is completed. □

Proof of Proposition 2

We first prove the first part. From Equations (2) and (1), we have

$$\mathbb{E}[\lambda | \mathcal{I}_i] = \frac{1}{\beta} \left[\left(1 - \frac{(L_S - 1)q}{(L_S - 1)q + 1} \frac{L_S}{L_S - 1} \right) \frac{\sigma_0^2}{\sigma_0^2 + \frac{\sigma_\omega^2}{\beta^2}} + \frac{(L_S - 1)q}{(L_S - 1)q + 1} \frac{L_S}{L_S - 1} \frac{1}{L_S} \frac{\sigma_0^2}{\sigma_0^2 + \frac{L_N^2 \sigma_\omega^2}{L_S^2 \beta^2}} \right]$$

and

$$\mathbb{E}[\lambda (M_S - 1) | \mathcal{I}_i] = \frac{(L_S - 1)q}{(L_S - 1)q + 1} \frac{L_S}{L_S - 1} \frac{L_S - 1}{L_S \beta} \frac{\sigma_0^2}{\sigma_0^2 + \frac{L_N^2 \sigma_\omega^2}{L_S^2 \beta^2}}.$$

Thus, solving (4) and using the above two relations yields

$$x_i = \frac{1 - \frac{(L_S - 1)q}{(L_S - 1)q + 1} \frac{\sigma_0^2}{\sigma_0^2 + \frac{L_N^2 \sigma_\omega^2}{L_S^2 \beta^2}}}{\frac{2}{\beta} \left[\left(1 - \frac{(L_S - 1)q}{(L_S - 1)q + 1} \frac{L_S}{L_S - 1} \right) \frac{\sigma_0^2}{\sigma_0^2 + \frac{\sigma_\omega^2}{\beta^2}} + \frac{(L_S - 1)q}{(L_S - 1)q + 1} \frac{1}{L_S - 1} \frac{\sigma_0^2}{\sigma_0^2 + \frac{L_N^2 \sigma_\omega^2}{L_S^2 \beta^2}} \right]} (\theta - \theta_0).$$

Matching the coefficient in the last equality with the conjectured one gives

$$\beta_m^* = \frac{1 - \frac{(L_S - 1)q}{(L_S - 1)q + 1} \frac{\sigma_0^2}{\sigma_0^2 + \frac{L_N^2 \sigma_\omega^2}{L_S^2 (\beta_m^*)^2}}}{\frac{2}{\beta_m^*} \left[\left(1 - \frac{(L_S - 1)q}{(L_S - 1)q + 1} \frac{L_S}{L_S - 1} \right) \frac{\sigma_0^2}{\sigma_0^2 + \frac{\sigma_\omega^2}{(\beta_m^*)^2}} + \frac{(L_S - 1)q}{(L_S - 1)q + 1} \frac{1}{L_S - 1} \frac{\sigma_0^2}{\sigma_0^2 + \frac{L_N^2 \sigma_\omega^2}{L_S^2 (\beta_m^*)^2}} \right]}.$$

Rearranging terms leads to

$$\frac{(L_S+1)q}{(L_S-1)q+1}\frac{\sigma_0^2}{\sigma_0^2+\frac{L_N^2\sigma_\omega^2}{L_S^2(\beta_m^*)^2}}+\frac{1-q}{(L_S-1)q+1}\frac{2\sigma_0^2}{\sigma_0^2+\frac{\sigma_\omega^2}{(\beta_m^*)^2}}=1, \tag{14}$$

which is further equivalent to

$$\frac{(L_S+1)q}{(L_S-1)q+1}\sigma_0^2\left(\sigma_0^2+\frac{\sigma_\omega^2}{(\beta_m^*)^2}\right)+2\sigma_0^2\frac{1-q}{(L_S-1)q+1}\left(\sigma_0^2+\frac{L_N^2\sigma_\omega^2}{L_S^2(\beta_m^*)^2}\right)$$
$$=\left(\sigma_0^2+\frac{\sigma_\omega^2}{(\beta_m^*)^2}\right)\left(\sigma_0^2+\frac{L_N^2\sigma_\omega^2}{L_S^2(\beta_m^*)^2}\right). \tag{15}$$

Let $t=\frac{\sigma_\omega^2}{\sigma_0^2(\beta_m^*)^2}$ for notational simplicity. Then, Equation (15) can be rewritten as

$$\begin{aligned}f(t):=&-\frac{L_N^2}{L_S^2}t^2+\left(\frac{(L_S+1)q}{(L_S-1)q+1}+\frac{2(1-q)}{(L_S-1)q+1}\frac{L_N^2}{L_S^2}-1-\frac{L_N^2}{L_S^2}\right)t\\&+\frac{2(1-q)+(L_S+1)q}{(L_S-1)q+1}-1\\=&-\frac{L_N^2}{L_S^2}t^2+\left(\frac{L_N^2}{L_S^2}\frac{1-q-L_Sq}{(L_S-1)q+1}+\frac{2q-1}{(L_S-1)q+1}\right)t+\frac{1}{(L_S-1)q+1}=0.\end{aligned} \tag{16}$$

Since the quadratic function satisfies $f(0)>0$ and $f(t)\to-\infty$ as $t\to\infty$, there exists a unique positive root to $f(t)=0$, given by

$$t=\frac{L_S^2}{2L_N^2}\left(\frac{\frac{L_N^2}{L_S^2}(1-q-L_Sq)+2q-1}{(L_S-1)q+1}+\sqrt{\left(\frac{\frac{L_N^2}{L_S^2}(1-q-L_Sq)+2q-1}{(L_S-1)q+1}\right)^2+\frac{4\frac{L_N^2}{L_S^2}}{(L_S-1)q+1}}\right).$$

Next, we show the inequalities comparing β_m^* with β_b^* for $a\in\{0,1\}$. Consider the first case $L_S>L_N^2$. Proposition 1 implies that $\sqrt{\frac{L_N^2\sigma_\omega^2}{L_S\sigma_0^2}}=\beta_b^*|_{a=1}<\beta_b^*|_{a=0}=\sqrt{\frac{\sigma_\omega^2}{\sigma_0^2}}$. Recall that $t=\frac{\sigma_\omega^2}{\sigma_0^2(\beta_m^*)^2}$. The relation $\beta_b^*|_{a=1}<\beta_m^*<\beta_b^*|_{a=0}$ is equivalent to $1<t<L_S/L_N^2$, which further translates to $f(L_S/L_N^2)<0<f(1)$, where $f(\cdot)$ is defined in (16). Simple calculations confirm that $f(L_S/L_N^2)<0<f(1)$ holds under the condition $L_S>L_N^2$. The proof for the case $L_S<L_N^2$ is analogous and omitted here. The proof is completed. □

Proof of Lemma 1

Recalling the notation $t=\frac{\sigma_\omega^2}{\sigma_0^2(\beta_m^*)^2}$ introduced in the proof of Proposition 2, we analyze the equilibrium for the following three cases of large markets.

- Scenario 1. Liquidity driven by informed trading: $L_S \to \infty$ and L_N is fixed. Defining $s = \frac{\sigma_\omega^2}{\sigma_0^2 (L_S \beta_m^*)^2}$, we express t as $t = sL_S^2$. From Equation (16), which defines t, we derive that s satisfies the following equation

$$s^2 - \frac{(1-q-L_S q)L_N^2 + (2q-1)L_S^2}{L_N^2 L_S^2[(L_S-1)q+1]} s - \frac{1}{L_N^2 L_S^2[(L_S-1)q+1]} = 0.$$

Solving for s, we obtain

$$s = \frac{(1-q-L_S q)L_N^2 + (2q-1)L_S^2 + \sqrt{\left[(1-q-L_S q)L_N^2 + (2q-1)L_S^2\right]^2 + 4L_N^2 L_S^2[(L_S-1)q+1]}}{2L_N^2 L_S^2[(L_S-1)q+1]}. \tag{17}$$

In this scenario, when $q > \frac{1}{2}$, it follows from (17) that $sL_S \to \frac{2q-1}{L_N^2 q}$, implying that $\sqrt{L_S}\beta_m^* \to L_N \sqrt{\frac{q}{2q-1}\frac{\sigma_\omega^2}{\sigma_0^2}}$. For $q < \frac{1}{2}$, from (16), t satisfies

$$t^2 - \frac{L_N^2(1-q-L_S q) + (2q-1)L_S^2}{L_N^2[(L_S-1)q+1]} t - \frac{L_S^2}{L_N^2[(L_S-1)q+1]} = 0. \tag{18}$$

A contradiction argument for (18) shows that $t \to \frac{1}{1-2q}$, leading to $\beta_m^* \to \sqrt{\frac{\sigma_\omega^2(1-2q)}{\sigma_0^2}}$.

- Scenario 2. Liquidity driven by noise trading: $L_S \to \infty$ and $\frac{L_N^2}{L_S} \to \infty$. From (18), we derive

$$t^2 - \frac{\frac{L_N^2}{L_S}(1-q-L_S q) + (2q-1)L_S}{\frac{L_N^2}{L_S}[(L_S-1)q+1]} t - \frac{L_S}{\frac{L_N^2}{L_S}[(L_S-1)q+1]} = 0. \tag{19}$$

In this scenario, we can show that $tL_N^2/L_S \to 1/q$. As a result, it holds that $\frac{\sqrt{L_S}}{L_N}\beta_m^* \to \sqrt{\frac{\sigma_\omega^2 q}{\sigma_0^2}}$.

- Scenario 3. "Middle" scenario: $L_S \to \infty$ and $\frac{L_N^2}{L_S} \to \rho$ for some $\rho \in (0, \infty)$. In this scenario, taking the limit of (19), we obtain

$$t^2 + \frac{\rho q - 2q + 1}{\rho q} t - \frac{1}{\rho q} = 0.$$

Solving for t, we find that the limit of t equals $\frac{-(\rho-2)q-1+\sqrt{((\rho-2)q+1)^2+4\rho q}}{2\rho q}$. Consequently, β_m^* converges to the specified expression in the lemma.

The proof is completed. □

Proof of Proposition 3

From Equation (5) for $\beta^* = \beta_b^*$ and Equation (7), we derive the expected trading volumes

$$TV_b = (qL_S\beta_b^*|_{a=1} + (1-q)\beta_b^*|_{a=0})\sqrt{\frac{2}{\pi}\sigma_0^2} + ((L_N-1)q+1)\sqrt{\frac{2}{\pi}\sigma_\omega^2},$$
$$TV_m = ((L_S-1)q+1)\beta_m^*\sqrt{\frac{2}{\pi}\sigma_0^2} + ((L_N-1)q+1)\sqrt{\frac{2}{\pi}\sigma_\omega^2}. \tag{20}$$

It follows that $TV_m > TV_b$ if and only if

$$(L_S q + 1 - q)\beta_m^* > qL_S\beta_b^*|_{a=1} + (1-q)\beta_b^*|_{a=0} = (qL_N\sqrt{L_S} + 1 - q)\sqrt{\frac{\sigma_\omega^2}{\sigma_0^2}},$$

where the equality follows from Proposition 1. Rearranging, this condition is equivalent to $t < \frac{(L_S q+1-q)^2}{(qL_N\sqrt{L_S}+1-q)^2}$, where $t = \frac{\sigma_\omega^2}{\sigma_0^2(\beta_m^*)^2}$ is as defined in the proof of Proposition 2. Observing that the quadratic equation $f(\cdot)$ in (16) has a unique positive root and satisfies that $f(0) > 0$ and $f(t) \to -\infty$ as $t \to \infty$, the inequality $t < \frac{(L_S q+1-q)^2}{(qL_N\sqrt{L_S}+1-q)^2}$ holds if and only if $f\left(\frac{(L_S q+1-q)^2}{(qL_N\sqrt{L_S}+1-q)^2}\right) < 0$. Expanding this condition leads to the inequality

$$\frac{L_N^2}{L_S^2}\frac{(L_S q+1-q)^4}{(qL_N\sqrt{L_S}+1-q)^4} - \left(\frac{L_N^2}{L_S^2}\frac{1-q-L_S q}{(L_S-1)q+1} + \frac{2q-1}{(L_S-1)q+1}\right)\frac{(L_S q+1-q)^2}{(qL_N\sqrt{L_S}+1-q)^2}$$
$$- \frac{1}{(L_S-1)q+1} > 0.$$

The preceding inequality can be equivalently written as

$$L_N^2(L_S q+1-q)^4((L_S-1)q+1) - L_S^2(qL_N\sqrt{L_S}+1-q)^4$$
$$- \left(L_N^2(1-q-L_S q) + (2q-1)L_S^2\right)(qL_N\sqrt{L_S}+1-q)^2(qL_S+1-q)^2 > 0. \tag{21}$$

To simplify notation, let $c = L_N/\sqrt{L_S}$. Then the left-hand side of (21) can be written as

$$(L_S q+1-q)^5c^2 - \left((1-q-L_S q)c^2 + (2q-1)L_S\right)(L_S qc+1-q)^2(L_S q+1-q)^2$$
$$- L_S(L_S qc+1-q)^4 =: A.$$

We next analyze the sign of A. After algebraic manipulation, we obtain

$$\frac{A}{L_S q(c-1)} = 2 + 2c - 2c^2 - 2L_S$$
$$+ q\left(-8 - 8c + 8c^2 + 10L_S + 8cL_S - 3c^2L_S - c^3L_S - L_S^2 - 5cL_S^2\right)$$

$$
\begin{aligned}
&+ q^2\Big(12 + 12c - 12c^2 - 18L_S - 24cL_S + 9c^2L_S + 3c^3L_S + 4L_S^2 \\
&\quad + 20cL_S^2 + c^2L_S^2 - c^3L_S^2 - 2cL_S^3 - 4c^2L_S^3\Big) \\
&+ q^3\Big(- 8 - 8c + 8c^2 + 14L_S + 24cL_S - 9c^2L_S - 3c^3L_S - 5L_S^2 \\
&\quad - 25cL_S^2 - 2c^2L_S^2 + 2c^3L_S^2 + 6cL_S^3 + 7c^2L_S^3 + c^3L_S^3 - c^2L_S^4 - c^3L_S^4\Big) \\
&+ q^4\Big(2 + 2c - 2c^2 - 4L_S - 8cL_S + 3c^2L_S + c^3L_S + 2L_S^2 + 10cL_S^2 \\
&\quad + c^2L_S^2 - c^3L_S^2 - 4cL_S^3 - 3c^2L_S^3 - c^3L_S^3 + c^2L_S^4 + c^3L_S^4\Big) \\
&=: b_0 + b_1q + b_2q^2 + b_3q^3 + b_4q^4 =: g(q).
\end{aligned}
$$

We have $g(1) = 0$,

$$
\begin{aligned}
g'(q)_{|q=1} &= -8 - 8c + 8c^2 + 10L_S + 8cL_S - 3c^2L_S - c^3L_S - L_S^2 - 5cL_S^2 \\
&+ 2\Big(12 + 12c - 12c^2 - 18L_S - 24cL_S + 9c^2L_S + 3c^3L_S + 4L_S^2 \\
&\quad + 20cL_S^2 + c^2L_S^2 - c^3L_S^2 - 2cL_S^3 - 4c^2L_S^3\Big) \\
&+ 3\Big(- 8 - 8c + 8c^2 + 14L_S + 24cL_S - 9c^2L_S - 3c^3L_S - 5L_S^2 \\
&\quad - 25cL_S^2 - 2c^2L_S^2 + 2c^3L_S^2 + 6cL_S^3 + 7c^2L_S^3 + c^3L_S^3 - c^2L_S^4 - c^3L_S^4\Big) \\
&+ 4\Big(2 + 2c - 2c^2 - 4L_S - 8cL_S + 3c^2L_S + c^3L_S + 2L_S^2 + 10cL_S^2 \\
&\quad + c^2L_S^2 - c^3L_S^2 - 4cL_S^3 - 3c^2L_S^3 - c^3L_S^3 + c^2L_S^4 + c^3L_S^4\Big) \\
&= -2cL_S^3 + c^2L_S^3 - c^3L_S^3 + c^2L_S^4 + c^3L_S^4 \\
&= cL_S^3(-2 + c - c^2 + cL_S + c^2L_S).
\end{aligned}
$$

Recall $c = L_N/\sqrt{L_S}$. As a result,

$$
-2 + c - c^2 + cL_S + c^2L_S = -2 + L_N/\sqrt{L_S} - L_N^2/L_S + L_N\sqrt{L_S} + L_N^2,
$$

which is positive for any $L_S \geq 2$ and $L_N \geq 1$. That is, $g'(q)_{|q=1} > 0$ for any $L_S \geq 2$ and $L_N \geq 1$. With some calculations we can also show that

$$
g(0) = b_0 = 2(1 + c - c^2 - L_S) = 2(1 + L_N/\sqrt{L_S} - L_N^2/L_S - L_S)
$$

is negative for any $L_S \geq 2$ and $L_N \geq 1$. Moreover, the coefficients of $g(\cdot)$ satisfy $b_1 < 0$, $b_2 < 0$, $b_3 < 0$ and $b_4 > 0$ for every $L_S \geq 4$ and $L_N \geq 4$.

We now analyze the number of positive roots of the polynomial $g(q) = b_0 + b_1 q + b_2 q^2 + b_3 q^3 + b_4 q^4 = 0$. Given that $b_0 < 0$, $b_1 < 0$, $b_2 < 0$, $b_3 < 0$ and $b_4 > 0$, the sequence of coefficients exhibits exactly one sign change. Consequently, by *Descartes' Rule of Signs*, the polynomial $g(\cdot)$ has exactly one positive root, counting multiplicities. Furthermore, we have $g(0) < 0$, $g(1) = 0$ and $g'(1) > 0$. These conditions imply that $g(q) < 0$ for any $0 < q < 1$.

Thus, when $L_S \geq 4$ and $L_N \geq 4$, it holds that $A > 0$ if $c < 1$, and $A < 0$ if $c > 1$. In other words, $TV_m > TV_b$ when $L_S > L_N^2$, and $TV_m < TV_b$ when $L_S < L_N^2$. The proof is completed. $\square$

Proof of Proposition 4

From Equation (9) for $\beta^* = \beta_m^*$ and Equation (11), we observe that $\mathbb{E}[PI_m] < \mathbb{E}[PI_b]$ if and only if

$$\sigma_0^2 \left[\frac{1-q}{\sigma_0^2 + \frac{\sigma_\omega^2}{(\beta_m^*)^2}} + \frac{q}{\sigma_0^2 + \frac{L_N^2 \sigma_\omega^2}{L_S^2 (\beta_m^*)^2}} \right] < \frac{1-q}{2} + \frac{q L_S}{L_S + 1},$$

which can be rewritten as

$$\frac{1-q}{1 + \frac{\sigma_\omega^2}{\sigma_0^2 (\beta_m^*)^2}} + \frac{q}{1 + \frac{L_N^2 \sigma_\omega^2}{\sigma_0^2 L_S^2 (\beta_m^*)^2}} < \frac{1-q}{2} + \frac{q L_S}{L_S + 1}.$$

Recall the notation $t = \frac{\sigma_\omega^2}{\sigma_0^2 (\beta_m^*)^2}$ introduced in the proof of Proposition 2. Substituting t into the inequality and rearranging the terms yields

$$\begin{aligned} & -\frac{L_N^2}{L_S^2} \left(\frac{1-q}{2} + \frac{q L_S}{L_S + 1} \right) t^2 \\ & + \left[\frac{(1-q) L_N^2}{L_S^2} + q - \left(\frac{1-q}{2} + \frac{q L_S}{L_S + 1} \right) \left(\frac{L_N^2}{L_S^2} + 1 \right) \right] t + 1 - \frac{1-q}{2} - \frac{q L_S}{L_S + 1} < 0. \end{aligned}$$

From (16), we can further simplify the inequality to

$$\begin{aligned} & -\left(\frac{1-q}{2} + \frac{q L_S}{L_S + 1} \right) \left[\left(\frac{L_N^2}{L_S^2} \frac{1 - q - L_S q}{(L_S - 1) q + 1} + \frac{2q - 1}{(L_S - 1) q + 1} \right) t + \frac{1}{(L_S - 1) q + 1} \right] \\ & + \left[\frac{(1-q) L_N^2}{L_S^2} + q - \left(\frac{1-q}{2} + \frac{q L_S}{L_S + 1} \right) \left(\frac{L_N^2}{L_S^2} + 1 \right) \right] t + 1 - \frac{1-q}{2} - \frac{q L_S}{L_S + 1} < 0. \end{aligned}$$

This can be rearranged as

$$\left[\frac{(1-q) L_N^2}{L_S^2} + q - \left(\frac{1-q}{2} + \frac{q L_S}{L_S + 1} \right) \left(\frac{2 L_N^2}{L_S^2} \frac{1-q}{(L_S - 1) q + 1} + \frac{(L_S + 1) q}{(L_S - 1) q + 1} \right) \right] t$$

$$+1-\left(\frac{1-q}{2}+\frac{qL_S}{L_S+1}\right)\frac{(L_S-1)q+2}{(L_S-1)q+1}=:k_1(q)t+k_2(q)<0.$$

Observe that

$$\begin{aligned}k_2(q)&=1-\frac{L_S+1+(L_S-1)q}{2(L_S+1)}\frac{(L_S-1)q+2}{(L_S-1)q+1}\\&\propto 2(L_S+1)[(L_S-1)q+1]-[L_S+1+(L_S-1)q][(L_S-1)q+2]\\&=(L_S-1)^2q(1-q)>0,\end{aligned}\tag{22}$$

and

$$\begin{aligned}&k_1(q)+k_2(q)\\&=\frac{(1-q)L_N^2}{L_S^2}+q-\left(\frac{1-q}{2}+\frac{qL_S}{L_S+1}\right)\left(\frac{2L_N^2}{L_S^2}\frac{1-q}{(L_S-1)q+1}+\frac{(L_S+1)q}{(L_S-1)q+1}\right)\\&\quad+1-\left(\frac{1-q}{2}+\frac{qL_S}{L_S+1}\right)\frac{(L_S-1)q+2}{(L_S-1)q+1}\\&=\frac{(1-q)L_N^2}{L_S^2}+1+q-2\left(\frac{1-q}{2}+\frac{qL_S}{L_S+1}\right)\frac{L_Sq+1+L_N^2(1-q)/L_S^2}{(L_S-1)q+1}\\&\propto(1-q)L_N^2(L_S+1)[(L_S-1)q+1]+(1+q)L_S^2(L_S+1)[(L_S-1)q+1]\\&\quad-[L_S(1+q)+1-q]L_S^2[L_Sq+1+L_N^2(1-q)/L_S^2]\\&=[(L_S^2-L_N^2)q+L_N^2+L_S^2][(L_S^2-1)q+L_S+1]\\&\quad-[(L_S-1)q+L_S+1][(L_S^3-L_N^2)q+L_S^2+L_N^2]\\&=-L_S(L_S-1)(L_S-L_N^2)q(1-q).\end{aligned}\tag{23}$$

Note that we have already shown that $k_2(q)>0$ for any $0<q<1$ in (22), $t>1$ if $L_S>L_N^2$, while $t<1$ if $L_S<L_N^2$ in the proof of Proposition 2. Thus, when $L_S>L_N^2$, it holds that $k_1(q)<0$ due to $k_1(q)+k_2(q)<0$ by (23), and consequently, $k_1(q)t+k_2(q)<k_1(q)+k_2(q)<0$. Similarly, when $L_S<L_N^2$, it holds that $k_1(q)t+k_2(q)>0$ if $k_1(q)\geq 0$, and $k_1(q)t+k_2(q)>k_1(q)+k_2(q)>0$ if $k_1(q)<0$ according to (23). Thus, we conclude that $\mathbb{E}[PI_m]<\mathbb{E}[PI_b]$ when $L_S>L_N^2$, and $\mathbb{E}[PI_m]>\mathbb{E}[PI_b]$ when $L_S<L_N^2$. The proof is completed. □

Proof of Proposition 5

By Equation (5) and $\mathbb{E}[M_N]=C_N$, we obtain

$$TV=\mathbb{E}[M_S\beta^*]\sqrt{\frac{2}{\pi}\sigma_0^2}+C_N\sqrt{\frac{2}{\pi}\sigma_\omega^2}.$$

Therefore, TV_b is equivalent to $\mathbb{E}[M_S\beta_b^*]$ in the benchmark case, while TV_m is equivalent to β_m^* in the main case due to the relation $\mathbb{E}[M_S\beta_m^*] = \mathbb{E}[M_S]\beta_m^* = C_S\beta_m^*$.

We first show that $\mathbb{E}[M_S\beta_b^*]$ increases (decreases) as q decreases when $C_N \geq 2$ ($C_N = 1$). To show this, by Proposition 1, we have

$$\mathbb{E}[M_S\beta_b^*] = \left[1 - q + q\sqrt{1 + \frac{C_S - 1}{q}}\left(1 + \frac{C_N - 1}{q}\right)\right]\sqrt{\frac{\sigma_\omega^2}{\sigma_0^2}}.$$

Differentiating it with respect to q, we get

$$\left(\sqrt{\frac{\sigma_\omega^2}{\sigma_0^2}}\right)^{-1}\frac{\partial\mathbb{E}[M_S\beta_b^*]}{\partial q} = -1 + \sqrt{1 + \frac{C_S - 1}{q}} - \frac{C_S - 1}{2q^2}\frac{q + C_N - 1}{\sqrt{1 + \frac{C_S - 1}{q}}}.$$

TV_b increases as q decreases if and only if $\frac{\partial\mathbb{E}[M_S\beta_b^*]}{\partial q} < 0$, which is equivalent to the condition

$$\tilde{L}_S - \frac{(\tilde{L}_S - 1)\tilde{L}_N}{2} < \sqrt{\tilde{L}_S}.$$

The above inequality holds if and only if $\tilde{L}_N \geq 2$, but does not hold when $\tilde{L}_N = 1$.

Next, we show that β_m^* increases (decreases) as q decreases when $C_N < C_S$ ($C_N > C_S$), and remains unchanged when $C_N = C_S$. Using the notation $t = \frac{\sigma_\omega^2}{\sigma_0^2(\beta_m^*)^2}$ introduced in the proof of Proposition 2, it is equivalent to show that t decreases (increases) as q decreases when $C_N < C_S$ ($C_N > C_S$), and remains constant when $C_N = C_S$. By the relations $(\tilde{L}_S - 1)q + 1 = C_S$, and $1 - q - \tilde{L}_S q = -(C_S + 2q - 2)$, (16) simplifies to

$$-\left[\frac{1 + (C_N - 1)/q}{1 + (C_S - 1)/q}\right]^2 t^2 + \left(\left[\frac{1 + (C_N - 1)/q}{1 + (C_S - 1)/q}\right]^2\frac{-(C_S + 2q - 2)}{C_S} + \frac{2q - 1}{C_S}\right)t + \frac{1}{C_S} = 0.$$

Defining $B(q) = \left[\frac{1+(C_N-1)/q}{1+(C_S-1)/q}\right]^2$, we can rewrite the above equation as

$$w(q,t) := B(q)t^2 + \left(B(q)\frac{C_S + 2q - 2}{C_S} + \frac{1 - 2q}{C_S}\right)t - \frac{1}{C_S} = 0. \tag{24}$$

Let t_{max} denote the unique positive root of Equation (24). Fixing $t = t_{max}$ and varying q, we next analyze changes in the left-hand side of (24). With some simple calculations, we obtain

$$\frac{\partial B(q)}{\partial q} = 2\frac{(1 + (C_N - 1)/q)(C_S - C_N)}{(1 + (C_S - 1)/q)^3 q^2}, 1 - B(q) = \frac{(2 + (C_N + C_S - 2)/q)(C_S - C_N)/q}{(1 + (C_S - 1)/q)^2}.$$

Taking the partial derivative of the left-hand side of (24) with respect to q, we derive

$$\frac{\partial B(q)}{\partial q}t_{max}^2 + \left(\frac{\partial B(q)}{\partial q}\frac{C_S + 2q - 2}{C_S} + (-1 + B(q))\frac{2}{C_S}\right)t_{max}$$

$$
\begin{aligned}
&\propto \frac{\partial B(q)}{\partial q}\left(t_{max}+\frac{C_S+2q-2}{C_S}\right)+(-1+B(q))\frac{2}{C_S}\\
&\propto (C_S-C_N)\Bigg[\left(1+\frac{C_N-1}{q}\right)\left(t_{max}+\frac{C_S+2q-2}{C_S}\right)\\
&\qquad -\left(2+\frac{C_N+C_S-2}{q}\right)\left(1+\frac{C_S-1}{q}\right)\frac{q}{C_S}\Bigg]\\
&=: (C_S-C_N)D(q). \qquad (25)
\end{aligned}
$$

By (25) and the fact that the quadratic equation $f(\cdot)$ in (16) has a unique positive root and satisfies $f(0)>0$ and $f(t)\to-\infty$ as $t\to\infty$, the conclusion is equivalent to the condition of $D(q)<0$ for any $q\in(0,1)$. Observe that $D(q)<0$ if and only if

$$
\begin{aligned}
t_{max} &< \frac{(2+(C_N+C_S-2)/q)(1+(C_S-1)/q)q-(1+(C_N-1)/q)(C_S+2q-2)}{(1+(C_N-1)/q)C_S}\\
&= \frac{(1+(C_S-1)/q)^2q+(1+(C_N-1)/q)(1-q)}{(1+(C_N-1)/q)C_S}\\
&= \frac{\tilde{L}_S^2q+\tilde{L}_N(1-q)}{\tilde{L}_N(\tilde{L}_Sq+1-q)} =: H(q).
\end{aligned}
$$

In terms of (24), the preceding inequality is equivalent to

$$
w(q,H(q)) := \frac{\tilde{L}_N^2}{\tilde{L}_S^2}(H(q))^2+\left(\frac{\tilde{L}_N^2}{\tilde{L}_S^2}\frac{C_S+2q-2}{C_S}+\frac{1-2q}{C_S}\right)H(q)-\frac{1}{C_S}>0.
$$

Note that

$$
\begin{aligned}
H(q)+\frac{C_S+2q-2}{C_S} &= \frac{(1+(C_S-1)/q)^2q+(1+(C_N-1)/q)(C_S+q-1)}{(1+(C_N-1)/q)C_S}\\
&= \frac{\tilde{L}_S^2q+\tilde{L}_N\tilde{L}_Sq}{\tilde{L}_N(\tilde{L}_Sq+1-q)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&w(q,H(q))>0\\
&\iff \frac{\tilde{L}_N^2}{\tilde{L}_S^2}\frac{\tilde{L}_S^2q+\tilde{L}_N(1-q)}{\tilde{L}_N(\tilde{L}_Sq+1-q)}\frac{\tilde{L}_S^2q+\tilde{L}_N\tilde{L}_Sq}{\tilde{L}_N(\tilde{L}_Sq+1-q)}+\frac{1-2q}{\tilde{L}_Sq-q+1}\frac{\tilde{L}_S^2q+\tilde{L}_N(1-q)}{\tilde{L}_N(\tilde{L}_Sq+1-q)}-\frac{1}{\tilde{L}_Sq-q+1}>0\\
&\iff (\tilde{L}_S+\tilde{L}_N\tilde{)}[\tilde{L}_S^2q+\tilde{L}_N(1-q)]\tilde{L}_Nq+(1-2q)[\tilde{L}_S^2q+\tilde{L}_N(1-q)]\tilde{L}_S-\tilde{L}_S\tilde{L}_N(1-q+\tilde{L}_Sq)>0\\
&\iff [(\tilde{L}_S+\tilde{L}_N)\tilde{L}_N-2\tilde{L}_S]\tilde{L}_S(\tilde{L}_S^2-\tilde{L}_N)q\\
&\qquad +(\tilde{L}_S^2+\tilde{L}_N\tilde{L}_S)\tilde{L}_N^2+\tilde{L}_S^2(\tilde{L}_S^2-\tilde{L}_N)-2\tilde{L}_N\tilde{L}_S^2-\tilde{L}_N\tilde{L}_S^2(\tilde{L}_S-1)>0
\end{aligned}
$$

$$\iff [(\tilde{L}_S + \tilde{L}_N)\tilde{L}_N - 2\tilde{L}_S](\tilde{L}_S^2 - \tilde{L}_N)q + (\tilde{L}_S + \tilde{L}_N)\tilde{L}_N^2 + \tilde{L}_S^3 - 2\tilde{L}_N\tilde{L}_S - \tilde{L}_N\tilde{L}_S^2 > 0$$
$$\iff [(\tilde{L}_N - 2)q + 1]\tilde{L}_S^3 + (\tilde{L}_N^2 q - \tilde{L}_N)\tilde{L}_S^2 + (\tilde{L}_N - 2)\tilde{L}_N\tilde{L}_S(1-q) + \tilde{L}_N^3(1-q) > 0,$$

where the last inequality is indeed true. Finally, the other results in this proposition follow directly from Proposition 3. The proof is completed. □

Proof of Proposition 6

From Equation (11) and the definition of $\tilde{L}_S = 1 + (C_S - 1)/q$, we have

$$\mathbb{E}[PI_b] = \left(\frac{1}{2}(1-q) + \frac{\tilde{L}_S}{\tilde{L}_S + 1}q\right)\sigma_0^2 = \frac{(C_S+1)q + C_S - 1}{2(2q + C_S - 1)}\sigma_0^2,$$

which decreases as q decreases. Thus, the conclusions regarding the monotonicity of $\mathbb{E}[PI_b]$ with respect to q follow.

From Equation (14), β_m^* satisfies

$$\frac{\tilde{L}_S + 1}{(\tilde{L}_S - 1)q + 1}\frac{q\sigma_0^2}{\sigma_0^2 + \frac{\tilde{L}_N^2\sigma_\omega^2}{\tilde{L}_S^2(\beta_m^*)^2}} + \frac{2}{(\tilde{L}_S - 1)q + 1}\frac{(1-q)\sigma_0^2}{\sigma_0^2 + \frac{\sigma_\omega^2}{(\beta_m^*)^2}} = 1,$$

which can be rewritten as

$$\frac{2 + (C_S - 1)/q}{2}\frac{q\sigma_0^2}{\sigma_0^2 + \frac{\tilde{L}_N^2\sigma_\omega^2}{\tilde{L}_S^2(\beta_m^*(q))^2}} + \frac{(1-q)\sigma_0^2}{\sigma_0^2 + \frac{\sigma_\omega^2}{(\beta_m^*(q))^2}} = \frac{C_S}{2},$$

where $\beta_m^*(q)$ emphasizes the dependence of β_m^* on q. Substituting q with $\hat{q}$ (where $\hat{q} < q$) yields

$$\frac{2 + (C_S - 1)/\hat{q}}{2}\frac{\hat{q}\sigma_0^2}{\sigma_0^2 + \frac{\tilde{L}_N^2\sigma_\omega^2}{\tilde{L}_S^2(\beta_m^*(\hat{q}))^2}} + \frac{(1-\hat{q})\sigma_0^2}{\sigma_0^2 + \frac{\sigma_\omega^2}{(\beta_m^*(\hat{q}))^2}} = \frac{C_S}{2}.$$

Furthermore, it follows from (9) that

$$\mathbb{E}[PI_m(q)]/\sigma_0^2 = \frac{(1-q)\sigma_0^2}{\sigma_0^2 + \frac{\sigma_\omega^2}{(\beta_m^*(q))^2}} + \frac{q\sigma_0^2}{\sigma_0^2 + \frac{\tilde{L}_N^2\sigma_\omega^2}{\tilde{L}_S^2(\beta_m^*(q))^2}}, \tag{26}$$
$$\mathbb{E}[PI_m(\hat{q})]/\sigma_0^2 = \frac{(1-\hat{q})\sigma_0^2}{\sigma_0^2 + \frac{\sigma_\omega^2}{(\beta_m^*(\hat{q}))^2}} + \frac{\hat{q}\sigma_0^2}{\sigma_0^2 + \frac{\tilde{L}_N^2\sigma_\omega^2}{\tilde{L}_S^2(\beta_m^*(\hat{q}))^2}}.$$

Using these four equalities, we derive

$$
\begin{aligned}
&\mathbb{E}[PI_m(\hat{q})] - \mathbb{E}[PI_m(q)] \\
&\propto \left(\frac{2+(C_S-1)/q}{2} - 1\right) \frac{\sigma_0^2 q}{\sigma_0^2 + \frac{\tilde{L}_N^2 \sigma_\omega^2}{\tilde{L}_S^2 (\beta_m^*(q))^2}} - \left(\frac{2+(C_S-1)/\hat{q}}{2} - 1\right) \frac{\sigma_0^2 \hat{q}}{\sigma_0^2 + \frac{\tilde{L}_N^2 \sigma_\omega^2}{\tilde{L}_S^2 (\beta_m^*(\hat{q}))^2}} \\
&\propto \frac{\sigma_0^2}{\sigma_0^2 + \frac{\tilde{L}_N^2 \sigma_\omega^2}{\tilde{L}_S^2 (\beta_m^*(q))^2}} - \frac{\sigma_0^2}{\sigma_0^2 + \frac{\tilde{L}_N^2 \sigma_\omega^2}{\tilde{L}_S^2 (\beta_m^*(\hat{q}))^2}} \\
&\propto \frac{1}{1 + \left(\frac{1+(C_N-1)/q}{1+(C_S-1)/q}\right)^2 t(q)} - \frac{1}{1 + \left(\frac{1+(C_N-1)/\hat{q}}{1+(C_S-1)/\hat{q}}\right)^2 t(\hat{q})}.
\end{aligned} \tag{27}
$$

Here $t(q)$ highlights the dependence of t on q.

As shown in the proof of Proposition 5, $t(\hat{q}) > t(q)$ ($t(\hat{q}) < t(q)$) when $C_S < C_N$ ($C_S > C_N$). Additionally, observe that $\frac{1+(C_N-1)/q}{1+(C_S-1)/q}$ decreases (increases) with q when $C_S < C_N$ ($C_S > C_N$). Thus, from (27), we conclude that $\mathbb{E}[PI_m(\hat{q})] > \mathbb{E}[PI_m(q)]$ when $C_S < C_N$, $\mathbb{E}[PI_m(\hat{q})] < \mathbb{E}[PI_m(q)]$ when $C_S > C_N$, and $\mathbb{E}[PI_m(\hat{q})] = \mathbb{E}[PI_m(q)]$ when $C_S = C_N$. Thus, the conclusions regarding the monotonicity of $\mathbb{E}[PI_m]$ with respect to q follow. Finally, the other results in this proposition follow directly from Proposition 4. The proof is completed. □

Proof of Proposition 7

Part (i). We first show that TV_m increases with large L_S. From Equation (20), it suffices to show that $((L_S-1)q+1)\beta_m^*$ increases with large L_S. Let $\nu = \frac{\sigma_\omega^2}{\sigma_0^2((L_S-1)q+1)^2(\beta_m^*)^2}$ for notational convenience. Observe that $\nu = \frac{t}{((L_S-1)q+1)^2}$. Then, Equation (16) can be rewritten as

$$
-L_N^2((L_S-1)q+1)^3\nu^2 + \left(L_N^2(1-q-L_Sq) + (2q-1)L_S^2\right)\nu + \frac{L_S^2}{((L_S-1)q+1)^2} = 0. \tag{28}
$$

Taking the partial derivative with respect to L_S on both sides of (28) yields

$$
\begin{aligned}
&\Big[-2L_N^2((L_S-1)q+1)^3\nu + \left(L_N^2(1-q-L_Sq) + (2q-1)L_S^2\right)\Big]\frac{\partial \nu}{\partial L_S} \\
&\qquad - 3qL_N^2((L_S-1)q+1)^2\nu^2 + \left(-L_N^2 q + 2(2q-1)L_S\right)\nu + \frac{2L_S(1-q)}{((L_S-1)q+1)^3} = 0.
\end{aligned} \tag{29}
$$

First, it follows from (28) that the coefficient of $\frac{\partial \nu}{\partial L_S}$ in (29) is negative for any $0 < q < 1$. Second, for $0 < q < \frac{1}{2}$, Lemma 1 shows that $\beta_m^* \to \sqrt{\frac{\sigma_\omega^2(1-2q)}{\sigma_0^2}}$, and thus $\nu = \mathcal{O}(\frac{1}{L_S^2})$ as $L_S \to \infty$. Consequently, the sum of the last three terms in (29) is dominated by $2(2q-1)\mathcal{O}(\frac{1}{L_S})$, which is negative. Therefore, $\frac{\partial \nu}{\partial L_S}$ is negative for large L_S when $0 < q < \frac{1}{2}$. For $\frac{1}{2} < q < 1$, Lemma 1 shows that $\sqrt{L_S}\beta_m^* \to L_N\sqrt{\frac{q}{2q-1}\frac{\sigma_\omega^2}{\sigma_0^2}}$, and thus $\nu L_S \to \frac{2q-1}{q^3 L_N^2}$ as $L_S \to \infty$. Consequently, the sum of the last three terms in (29) is dominated by

$$-3q^3 L_N^2 \left(\frac{2q-1}{q^3 L_N^2}\right)^2 + 2(2q-1)\left(\frac{2q-1}{q^3 L_N^2}\right),$$

which is also negative. Thus, $\frac{\partial \nu}{\partial L_S} < 0$ for large L_S when $\frac{1}{2} < q < 1$. For $q = \frac{1}{2}$, from (28), we can show by contradiction that $L_S \nu \to 0$ and $L_S^3 \nu^2 \to \frac{1}{L_N^2 q^5}$ as $L_S \to \infty$. This implies that the sum of the last three terms in (29) is dominated by $-3q^2 L_N^2 \mathcal{O}(\frac{1}{L_S})$, which is negative. Recalling that the coefficient of $\frac{\partial \nu}{\partial L_S}$ in (29) is negative, we conclude that ν decreases and TV_m increases with large L_S for any $0 < q < 1$.

Next, we show the conclusion regarding expected price informativeness. From (26) and (14), we have

$$\mathbb{E}[PI_m]/\sigma_0^2 = \frac{(L_S-1)q+1}{L_S+1} + \left(1 - \frac{2}{L_S+1}\right)\frac{1-q}{1+t}, \tag{30}$$

where $t = \frac{\sigma_\omega^2}{\sigma_0^2(\beta_m^*)^2}$. Additionally, from (16), we have

$$-L_N^2 t^2 + \left(L_N^2 \frac{1-q-L_S q}{(L_S-1)q+1} + \frac{(2q-1)L_S^2}{(L_S-1)q+1}\right)t + \frac{L_S^2}{(L_S-1)q+1} = 0. \tag{31}$$

Taking the partial derivative with respect to L_S on both sides of (30) yields

$$\frac{\partial \mathbb{E}[PI_m]/\sigma_0^2}{\partial L_S} = \frac{2q-1}{(L_S+1)^2} + \frac{2(1-q)}{(L_S+1)^2(1+t)} - \frac{L_S-1}{L_S+1}\frac{1-q}{(1+t)^2}\frac{\partial t}{\partial L_S}. \tag{32}$$

Taking the partial derivative with respect to L_S on both sides of (31) gives

$$\begin{aligned}&\left(-2L_N^2 t + L_N^2\frac{1-q-L_S q}{(L_S-1)q+1} + \frac{(2q-1)L_S^2}{(L_S-1)q+1}\right)\frac{\partial t}{\partial L_S}\\ &+ \left(L_N^2\frac{-2q(1-q)}{((L_S-1)q+1)^2} + (2q-1)\frac{L_S^2 q + 2L_S(1-q)}{((L_S-1)q+1)^2}\right)t + \frac{L_S^2 q + 2L_S(1-q)}{((L_S-1)q+1)^2} = 0.\end{aligned} \tag{33}$$

We analyze the following three cases.

Case 1: Suppose $0 < q < \frac{1}{2}$. By Lemma 1, $\beta_m^* \to \sqrt{\frac{\sigma_\omega^2(1-2q)}{\sigma_0^2}}$, and thus $t \to \frac{1}{1-2q}$ as $L_S \to \infty$. Consequently, from (31), we have $(1 + (2q-1)t)L_S \to \frac{2q(1-q)L_N^2}{(1-2q)^2}$. It follows from (33) that

$$\left(-2L_N^2 t\frac{(L_S-1)q+1}{L_S^2} + L_N^2\frac{1-q-L_S q}{L_S^2} + 2q - 1\right)L_S^2\frac{\partial t}{\partial L_S}$$

$$+ L_N^2 t \frac{-2q(1-q)}{(L_S-1)q+1} + (1+(2q-1)t)L_S \frac{L_S^2 q + 2L_S(1-q)}{((L_S-1)q+1)L_S} = 0,$$

and thus as $L_S \to \infty$,

$$(1-2q)L_S^2 \frac{\partial t}{\partial L_S} \to \frac{2q(1-q)L_N^2}{(1-2q)^2}.$$

Combining this with the relation $2q - 1 + \frac{2(1-q)}{1+t} \to 0$ and (32), we conclude that $\frac{\partial \mathbb{E}[PI_m]/\sigma_0^2}{\partial L_S} < 0$ for large L_S.

Case 2: Suppose $\frac{1}{2} < q < 1$. By Lemma 1, $\sqrt{L_S}\beta_m^* \to L_N \sqrt{\frac{q}{2q-1}\frac{\sigma_\omega^2}{\sigma_0^2}}$, and thus $t/L_S \to \frac{2q-1}{qL_N^2}$ as $L_S \to \infty$. From (33), we conclude that $\frac{\partial t}{\partial L_S} \to \frac{2q-1}{qL_N^2}$. From (32), we observe that $\frac{\partial \mathbb{E}[PI_m]/\sigma_0^2}{\partial L_S}$ is dominated by

$$\frac{2q-1}{(L_S+1)^2} - \frac{L_S-1}{L_S+1}\frac{1-q}{(1+t)^2}\frac{\partial t}{\partial L_S} = (2q-1)\left(1 - \frac{q(1-q)L_N^2}{(2q-1)^2}\right)\mathcal{O}\left(\frac{1}{L_S^2}\right),$$

which is negative if $q(1-q)L_N^2 > (2q-1)^2$, or equivalently,

$$\frac{1}{2} < q < \frac{1}{2}\left(1 + \sqrt{\frac{L_N^2}{L_N^2+4}}\right),$$

and positive if $q(1-q)L_N^2 < (2q-1)^2$, or equivalently,

$$\frac{1}{2}\left(1 + \sqrt{\frac{L_N^2}{L_N^2+4}}\right) < q < 1.$$

Case 3: Suppose $q = \frac{1}{2}$. In this case, from (31), we have $t/\sqrt{L_S} \to \sqrt{2}/L_N$. It then follows from (33) that $t\frac{\partial t}{\partial L_S} \to 1/L_N^2$. Consequently, $\frac{\partial \mathbb{E}[PI_m]/\sigma_0^2}{\partial L_S}$ is dominated by the third term in (32), which is negative.

Part (ii). By Equations (20) and (30), to show that increasing L_N always increases TV_m and $\mathbb{E}[PI_m]$, it suffices to show that β_m^* increases, or equivalently, t decreases with L_N. To this end, from (31), we have

$$\left(-2L_N^2 t + L_N^2 \frac{1-q-L_S q}{(L_S-1)q+1} + \frac{(2q-1)L_S^2}{(L_S-1)q+1}\right)\frac{\partial t}{\partial L_N} - 2L_N t^2 + 2L_N \frac{1-q-L_S q}{(L_S-1)q+1} t = 0, \tag{34}$$

where the coefficient of $\frac{\partial t}{\partial L_N}$ in (34) is negative by (31). Moreover, from (31), we have

$$2L_N t^2 - 2L_N \frac{1-q-L_S q}{(L_S-1)q+1} t \propto 1 + (2q-1)t,$$

which is positive when $\frac{1}{2} \leq q < 1$. Additionally, when $0 < q < \frac{1}{2}$, it holds that $f(1/(1-2q)) < 0$ (where $f(\cdot)$ is given by (16)), implying $t < \frac{1}{1-2q}$. Thus, $1 + (2q-1)t > 0$, and consequently, it follows from (34) that $\partial t/\partial L_N < 0$ for any $0 < q < 1$. As a result, t decreases with L_N.

Part (iii). We first show that increasing q increases $\mathbb{E}[PI_m]$. From (30), we have

$$\begin{aligned}
\frac{\partial \mathbb{E}[PI_m]/\sigma_0^2}{\partial q} &= \frac{L_S-1}{L_S+1} + \frac{L_S-1}{L_S+1}\frac{-(1+t)-(1-q)\frac{\partial t}{\partial q}}{(1+t)^2} \\
&\propto 1 - \frac{1}{1+t} - \frac{1-q}{(1+t)^2}\frac{\partial t}{\partial q} \\
&\propto t - \frac{1-q}{1+t}\frac{\partial t}{\partial q}.
\end{aligned} \tag{35}$$

Moreover, from (31), we obtain

$$-L_N^2((L_S-1)q+1)t^2 + \left(L_N^2(1-(L_S+1)q) + (2q-1)L_S^2\right)t + L_S^2 = 0. \tag{36}$$

Taking the partial derivative with respect to q on both sides of (36) yields

$$\begin{aligned}
&\Big(-2L_N^2((L_S-1)q+1)t + L_N^2(1-(L_S+1)q) + (2q-1)L_S^2\Big)\frac{\partial t}{\partial q} \\
&\quad - L_N^2(L_S-1)t^2 - L_N^2(L_S+1)t + 2L_S^2 t = 0,
\end{aligned} \tag{37}$$

where the coefficient of $\frac{\partial t}{\partial q}$ in (37) is negative by (36). Combining (35) and (37), we obtain

$$\begin{aligned}
\frac{\partial \mathbb{E}[PI_m]/\sigma_0^2}{\partial q} &\propto t - \frac{1-q}{1+t}\frac{-L_N^2(L_S-1)t^2 - L_N^2(L_S+1)t + 2L_S^2 t}{2L_N^2((L_S-1)q+1)t - L_N^2(1-(L_S+1)q) - (2q-1)L_S^2} \\
&\propto (1+t)\Big(2L_N^2((L_S-1)q+1)t - L_N^2(1-(L_S+1)q) - (2q-1)L_S^2\Big) \\
&\qquad + (1-q)\Big(L_N^2(L_S-1)t + L_N^2(L_S+1) - 2L_S^2\Big) \\
&= 2L_N^2((L_S-1)q+1)t^2 + \Big(L_N^2 L_S(2q+1) - L_S^2(2q-1)\Big)t + L_N^2 L_S - L_S^2.
\end{aligned} \tag{38}$$

From (36), we have

$$L_N^2((L_S-1)q+1)t^2 = \left(L_N^2(1-(L_S+1)q) + (2q-1)L_S^2\right)t + L_S^2.$$

Substituting this into (38) yields

$$\begin{aligned}
\frac{\partial \mathbb{E}[PI_m]/\sigma_0^2}{\partial q} &\propto 2\left(L_N^2(1-(L_S+1)q) + (2q-1)L_S^2\right)t + 2L_S^2 \\
&\qquad + \Big(L_N^2 L_S(2q+1) - L_S^2(2q-1)\Big)t + L_N^2 L_S - L_S^2
\end{aligned}$$

$$= \Big((L_S + 2(1-q))L_N^2 + L_S^2(2q-1) \Big) t + L_N^2 L_S + L_S^2$$
$$= (L_S + 2(1-q))L_N^2 t + ((2q-1)t+1)L_S^2 + L_N^2 L_S.$$

This expression is positive because $(2q-1)t+1>0$, as proven in Part (ii) of this proposition.

We now show the second result in Part (iii). Recall the notation $\nu = \frac{\sigma_\omega^2}{\sigma_0^2((L_S-1)q+1)^2(\beta_m^*)^2}$. From (20), we have

$$TV_m = \sqrt{\frac{\sigma_\omega^2}{\sigma_0^2 \nu}} \sqrt{\frac{2}{\pi}\sigma_0^2} + ((L_N-1)q+1)\sqrt{\frac{2}{\pi}\sigma_\omega^2}$$
$$= \sqrt{\frac{2\sigma_\omega^2}{\pi}} \left(\frac{1}{\sqrt{\nu}} + (L_N-1)q+1 \right). \tag{39}$$

As a result,

$$\frac{\partial TV_m}{\partial q} = \sqrt{\frac{2\sigma_\omega^2}{\pi}} \left(-\frac{1}{2\nu^{\frac{3}{2}}}\frac{\partial \nu}{\partial q} + L_N \right). \tag{40}$$

Taking the partial derivative with respect to q on both sides of (28) yields

$$\Big[-2L_N^2((L_S-1)q+1)^3\nu + L_N^2(1-q-L_Sq) + (2q-1)L_S^2 \Big] \frac{\partial \nu}{\partial q}$$
$$-3L_N^2((L_S-1)q+1)^2(L_S-1)\nu^2 + \left(-L_N^2(L_S+1)+2L_S^2\right)\nu - \frac{2L_S^2(L_S-1)}{((L_S-1)q+1)^3} = 0. \tag{41}$$

We analyze the following three cases.

Case 1: Suppose $0<q<\frac{1}{2}$. By Lemma 1, $\beta_m^* \to \sqrt{\frac{\sigma_\omega^2(1-2q)}{\sigma_0^2}}$, and thus $\nu L_S^2 \to \frac{1}{q^2(1-2q)}$ as $L_S\to\infty$. It then follows from (41) that $L_S^2\frac{\partial\nu}{\partial q} \to -\frac{2(1-3q)}{q^3(1-2q)^2}$. Combining this with (40), we find that for large L_S, $-\frac{1}{2\nu^{\frac{3}{2}}}\frac{\partial\nu}{\partial q}+L_N$ is dominated by

$$\frac{1-3q}{(1-2q)^{\frac{1}{2}}}L_S.$$

Thus, for large L_S, $\frac{\partial TV_m}{\partial q}$ is positive when $0<q<\frac{1}{3}$, and negative when $\frac{1}{3}<q<\frac{1}{2}$.

Case 2: Suppose $\frac{1}{2}<q<1$. By Lemma 1, $\sqrt{L_S}\beta_m^* \to L_N\sqrt{\frac{q}{2q-1}\frac{\sigma_\omega^2}{\sigma_0^2}}$, and thus $\nu L_S \to \frac{2q-1}{q^3L_N^2}$ as $L_S\to\infty$. It then follows from (41) that $L_S\frac{\partial\nu}{\partial q} \to \frac{1}{q^3L_N^2}\left(-4+\frac{3}{q}\right)$. Combining this with (40), we find that for large L_S, $-\frac{1}{2\nu^{\frac{3}{2}}}\frac{\partial\nu}{\partial q}+L_N$ is dominated by

$$\frac{q^{\frac{3}{2}}\sqrt{L_S}}{2(2q-1)^{\frac{3}{2}}}\left(4-\frac{3}{q}\right).$$

Thus, for large L_S, $\frac{\partial TV_m}{\partial q}$ is positive when $\frac{3}{4} < q < 1$, and negative when $\frac{1}{2} < q < \frac{3}{4}$.

Case 3: Suppose $q = \frac{1}{2}$. From (31), $t/\sqrt{L_S} \to \sqrt{2}/L_N$, and $\nu L_S^{\frac{3}{2}} \to \frac{\sqrt{2}}{L_N q^2}$ as $L_S \to \infty$. It then follows from (41) that $L_S \frac{\partial \nu}{\partial q} \to \frac{1}{L_N^2 q^3}$. Combining this with (40), we find that for large L_S, $-\frac{1}{2\nu^{\frac{3}{2}}} \frac{\partial \nu}{\partial q} + L_N$ is dominated by

$$-\frac{L_S^{\frac{5}{4}}}{2^{\frac{7}{4}}\sqrt{L_N}},$$

which is negative.

We now show the last result in Part (iii). We analyze the following three cases.

Case 1: Suppose $(L_S+1)q > 1$. It follows from (28) that $\nu \to 0$ and $L_N^2 \nu \to \frac{L_S^2}{((L_S-1)q+1)^2((L_S+1)q-1)}$. Consequently, from (41), we have

$$L_N^2 \frac{\partial \nu}{\partial q} \to \frac{\frac{L_S^2(L_S+1)}{((L_S-1)q+1)^2((L_S+1)q-1)} + \frac{2L_S^2(L_S-1)}{((L_S-1)q+1)^3}}{1 - q - L_S q}.$$

Thus, by (39) and (40),

$$\frac{TV_m}{L_N} \to \sqrt{\frac{2\sigma_\omega^2}{\pi}} \left(\frac{((L_S-1)q+1)\sqrt{(L_S+1)q-1}}{L_S} + q \right).$$

As a result, $\frac{\partial TV_m}{\partial q}$ is dominated by

$$\sqrt{\frac{2\sigma_\omega^2}{\pi}} \left(\frac{1}{2} \frac{((L_S-1)q+1)^3((L_S+1)q-1)^{\frac{3}{2}}}{L_S^3} \frac{\frac{L_S^2(L_S+1)}{((L_S-1)q+1)^2((L_S+1)q-1)} + \frac{2L_S^2(L_S-1)}{((L_S-1)q+1)^3}}{(L_S+1)q-1} + 1 \right) L_N$$
$$= \sqrt{\frac{2\sigma_\omega^2}{\pi}} \left(\frac{1}{2} \frac{((L_S+1)q-1)^{\frac{1}{2}}}{L_S} \left(\frac{(L_S+1)((L_S-1)q+1)}{(L_S+1)q-1} + 2(L_S-1) \right) + 1 \right) L_N.$$

Case 2: Suppose $(L_S+1)q < 1$. From (28), $\nu \to \frac{1-(L_S+1)q}{((L_S-1)q+1)^3}$ as $L_N \to \infty$. Thus, by (39),

$$\frac{TV_m}{L_N} \to \sqrt{\frac{2\sigma_\omega^2}{\pi}} q.$$

Case 3: Suppose $(L_S+1)q = 1$. From (28), $L_N^2 \nu^2 \to \frac{L_S^2}{((L_S-1)q+1)^5}$. Thus, by (39),

$$TV_m \to \sqrt{\frac{2\sigma_\omega^2}{\pi}} \left(\frac{((L_S-1)q+1)^{\frac{5}{4}}}{\sqrt{L_S}} \sqrt{L_N} + qL_N \right),$$

and

$$\frac{TV_m}{L_N} \to \sqrt{\frac{2\sigma_\omega^2}{\pi}} q.$$

Thus, for large L_N, TV_m increases with $q \in (0, 1)$.

The proof is completed. □

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